Extended Markov numbers and integer geometry

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Abstract

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We call two lines intersecting at the point $(0,0)$ in $\mathbb{R}^2$ an *arrangement*. Arrangements are solutions to equations

$$ax^2 + bxy + cy^2 = 0,$$

for real a, b, and c satisfying $b^2 - 4ac > 0$. Solutions to the Markov Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz$$

are called *Markov numbers*, and are a key component in classifying 2-D arrangements up to $\mathrm{SL}(2,\mathbb{Z})$ equivalence, as shown by A. Markov in the late 19$^{\text{th}}$ century.

In this thesis we prove the following results:

(i) There is a relation called the *Perron identity* between infinite sequences of positive integers and the minima of binary quadratic forms. We find the geometric meaning behind this result, and extend it to all values of forms in Theorem 2.3.14;

(ii) Markov numbers may be described through combinations of the sequences $(1,1)$ and $(2,2)$. These combinations are known either as Cohn words or Christoffel words. We propose an extension to the theory of Markov numbers. The initial sequences used in the classical construction are substituted by new ones with certain properties. In Theorems 4.1.14, 4.1.27, and 5.1.5 we show that these sequences create systems of numbers similar to Markov numbers;

(iii) There is a conjecture first noted by G. Frobenius in 1913 that says triples of Markov numbers are defined by their largest element. We see analogues to this *uniqueness conjecture* for each extension of Markov numbers. We show that for certain extensions of Markov numbers, the generalised uniqueness conjecture fails in Theorem 5.2.8.

Declaration

No part of the work in this thesis has been submitted for a degree or other qualification at this or any other institution.

Material in Section 2.3 appears in the following paper.

[54] O. Karpenkov and M. Van Son. Generalized Perron Identity for broken lines. *Journal de théorie des nombres de Bordeaux*, 31:131–144, 2019.

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Maps between triple graphs

Chapter 1

Introduction

In this thesis we study the following problem. Let f be the form

$$f(x, y) = ax^2 + bxy + cy^2$$

with real coefficients and positive discriminant $\Delta(f) = b^2 - 4ac$.

Question. For such a form f, at what point in the integer lattice does f attain its least positive value? At what point in the integer lattice does f attain its greatest negative value?

In the late 19$^{\text{th}}$ century A. Markov [63, 64] found a method of defining classes of forms by triples of positive integers called *Markov triples*. These are solutions to the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz.$$

The numbers in Markov triples are called *regular Markov numbers*. Markov showed that the form defined by a Markov triple (a, b, c), with $a \leq c \leq b$, attains its least positive value at the point $(1, 0)$. The value at this point is $f(1, 0) = b$. Our objective in this thesis is to extend the notion of these Markov triples, by generating forms whose least positive value is attained at the point $(1, 0)$.

We begin this chapter in Section 1.1 with a history of regular Markov numbers. We also mention other contemporary approaches to generalising Markov numbers.

We continue in Section 1.2, where we present a motivating example to relate the study of regular Markov numbers with integer geometry.

Basic notions of integer geometry are introduced in Section 1.3.

Finally in Section 1.4 we present the layout and structure of the thesis.

1.1 History of regular Markov numbers

Let us begin with an overview of the central points of the theory of regular Markov numbers. This theory was introduced by A. Markov in his papers [63, 64]. Comprehensive introductions to the study of regular Markov numbers include the book by T. Cusick and M. Flahive [23], the book by J. W. S. Cassels [17, pp. 27–30], and the paper by A. Bombieri [10].

A good survey of the literature on regular Markov numbers was performed by A. V. Malyshev in [62]. Contemporary study of regular Markov numbers includes the relation to hyperbolic geometry, see for example the papers by C. Series [84] and B. Springborn [89]. The growth of regular Markov numbers has been studied by D. Zagier [97] and A. Baragar [3]. Other interesting works related to Markov theory include the papers by L. Tornheim [92] on Diophantine approximation, J. H. Silverman [86] on the Markov Diophantine equation over quadratic imaginary fields, and K. Spalding and A. P. Veselov [88] on Lyapunov exponents.

We recall Markov's theorem, Theorem 1.1.9, in Subsection 1.1.1. Proofs of Markov's theorem are given in Markov's papers themselves, and also in the aforementioned works of T. Cusick and M. Flahive [23] and A. Bombieri [10], the book by J. W. S. Cassels [16], and the book by M. Aigner [2].

Markov's theorem contains a result on the *Markov spectrum*, a spectrum of real numbers related to the minima of indefinite binary quadratic forms. This spectrum has been studied extensively, with particular attention to gaps in the spectrum above 3, as seen in the works of T. Cusick, M. Flahive, W. Moran, and A. D. Pollington [24, 25, 26, 23, 28].

In Subsection 1.1.2 we introduce the graph structure of regular Markov numbers. This structure forms part of the foundation of our work. We state the uniqueness conjecture for regular Markov numbers, which was first introduced by G. Frobenius [38] in 1913.

Finally in Subsection 1.1.3 we give a short history of recent developments in the study of Markov numbers.

1.1.1 Markov's theorem

In this section we introduce Markov's theorem and the uniqueness conjecture for regular Markov numbers. Let us start with a definition.

Definition 1.1.1. Let f be a binary quadratic form with real coefficients,

$$f(x, y) = ax^2 + bxy + cy^2,$$

and positive discriminant $\Delta(f) = b^2 - 4ac$ (i.e. an *indefinite form*). Define the

Markov minimum of f to be

$$m(f) = \inf_{\mathbb{Z}^2 \setminus \{(0,0)\}} |f|.$$

Note that the Markov minima of a form f and a form λf, where λ is a non zero integer, differ by the scaling factor $|\lambda|$. In order to study Markov minima we define a property of forms that does not change with scaling. We have the following definition.

Definition 1.1.2. Let the *Markov spectrum* be the set

$$\mathcal{M} = \left\{ \frac{\sqrt{\Delta(f)}}{m(f)} \ : \ f \text{ is an indefinite binary quadratic form} \right\}.$$

Remark 1.1.3.

(i) Unless otherwise stated we take the term *form* to mean *indefinite binary quadratic form*.

(ii) For a form f we refer to the value

$$\frac{\sqrt{\Delta(f)}}{m(f)}$$

as the *Markov value of* f.

(iii) Note that two forms f and λf, differing by a non zero scalar λ, have the same Markov value. Indeed, let $f(x, y) = ax^2+bxy+cy^2$. Then $m(\lambda f) = |\lambda| m(f)$ and

$$\sqrt{\Delta(\lambda f)} = \sqrt{\lambda^2 b^2 - 4\lambda^2 ac} = |\lambda| \sqrt{\Delta(f)}.$$

Recall the definition for $\mathrm{SL}(2, \mathbb{Z})$ equivalence of forms.

Definition 1.1.4. Two forms f and g are $\mathrm{SL}(2, \mathbb{Z})$ *equivalent* if there exist integers a, b, c, and d such that $ad - bc = \pm 1$ and

$$f(ax + by, cx + dy) = g(x, y).$$

Remark 1.1.5. It is clear from the theory of forms that equivalent forms have the same Markov value. We discuss this in detail in Section 2.2. Here and below we refer to $\mathrm{SL}(2, \mathbb{Z})$ equivalent forms simply as being *equivalent*.

Definition 1.1.6.

(i) The *Markov Diophantine equation* is

$$x^2 + y^2 + z^2 = 3xyz.$$

A positive solution (a, Q, b) of this equation is called a *Markov triple*. Elements of Markov triples are called *regular Markov numbers*. Unless otherwise stated, a Markov triple (a, Q, b) is ordered such that $a \leq Q$ and $b \leq Q$. We denote such a Markov triple by

$$Q_{a,b} = (a, Q, b).$$

(ii) For a Markov triple (a, Q, b) define the integer u be the least positive solution to the equivalence

$$xa \equiv \pm b \mod Q,$$

and let the integer v be defined by the equation

$$vQ = u^2 + 1.$$

From this we define the *Markov form associated with* $Q_{a,b}$ to be

$$f_{Q_{a,b}}(x, y) = Qx^2 + (3Q - 2u)xy + (v - 3u)y^2.$$

The following proposition may be proved using structure of Markov numbers that we introduce in the next subsection. A proof may be found in [23, 2].

Proposition 1.1.7. *Any two numbers in a Markov triple are coprime.*

Remark 1.1.8. The existence of u in Definition 1.1.6 (ii) follows from Proposition 1.1.7 for a Markov triple (a, Q, b) with $Q > 1$ (if $Q = 1$ then $u = 1$). Since

$$a^2 + b^2 + Q^2 = 3abQ$$

we have that a^2+b^2 is divisible by Q. Hence

$$a^2 \equiv -b^2 \mod Q.$$

As $\gcd(b, Q) = 1$ there is a positive integer $d < Q$ such that $bd \equiv 1 \mod Q$. Then

$$\begin{aligned} a^2 d &\equiv -b^2 d \mod Q \\ a(ad) &\equiv -b \mod Q \\ a(-ad) &\equiv b \mod Q. \end{aligned}$$

Hence there are two positive integers equivalent to $\pm ad \mod Q$ satisfying

$$ax \equiv \pm b \mod Q.$$

In his papers [63, 64] A. Markov set about answering the following question; at what non-zero integer point does a form f attain its Markov minimum? His partial answer to this question comes through the study of a discrete subset of the Markov spectrum. In his work he classified all forms associated to this subset.

The following is the theorem of Markov.

Theorem 1.1.9 (Markov's Theorem)**.**

(*i*) *The Markov spectrum below 3 is discrete, and given by the numbers*

$$\frac{\sqrt{9Q^2-4}}{Q},$$

where Q is a regular Markov number. The number 3 *is the first limiting point of the Markov spectrum.*

(*ii*) *Let $f_{Q_{a,b}}$ be the Markov form associated with $Q_{a,b}$. Then the Markov value of $f_{Q_{a,b}}$ is*

$$\frac{\sqrt{\Delta\big(f_{Q_{a,b}}\big)}}{m\big(f_{Q_{a,b}}\big)} = \frac{\sqrt{9Q^2-4}}{Q}.$$

(*iii*) *All forms with the same Markov value below three are equivalent. In particular, let g be a form with Markov value below 3. The Markov value of g is given by some regular Markov number Q, where*

$$\frac{\sqrt{\Delta(g)}}{m(g)} = \frac{\sqrt{9Q^2-4}}{Q}.$$

Then the form g is equivalent to the Markov form $f_{Q_{a,b}}$.

The proof of this theorem is given by Markov in his papers [63, 64]. More readable versions of the proof are given in [16, 2, 10].

Example 1. Table 1.1 shows the first ten regular Markov numbers, their Markov forms, and their Markov values.

We study the Markov spectrum in more detail in Chapter 2, specifically in Section 2.5.

1.1.2 Graph of regular Markov numbers

In this subsection we study regular Markov numbers in some more detail, and see their graph structure. Recall that a Markov triple (a, Q, b) has $a \le Q$ and $b \le Q$, and observe the following two facts, which are proven, for example see [23].

Proposition 1.1.10.

(*i*) *Any permutation of (a, Q, b) is also a solution to the Markov Diophantine equation.*

(*ii*) *Both $(a, 3Qa - b, Q)$ and $(Q, 3Qb - a, b)$ are Markov triples.*

Table 1.1: The first ten regular Markov numbers, Markov forms, and Markov values.

(a,Q,b)	$f_{Q_{a,b}}(x,y)$	$\dfrac{\sqrt{\Delta(f_{Q_{a,b}})}}{m(f_{Q_{a,b}})}$
$(1,1,1)$	x^2+xy-y^2	$\sqrt{5}$
$(1,2,1)$	$2x^2+4xy-2y^2$	$\sqrt{8}$
$(1,5,2)$	$5x^2+11xy-5y^2$	$\dfrac{\sqrt{221}}{5}$
$(1,13,5)$	$13x^2+29xy-13y^2$	$\dfrac{\sqrt{1517}}{13}$
$(5,29,2)$	$29x^2+63xy-31y^2$	$\dfrac{\sqrt{7565}}{29}$
$(1,34,13)$	$34x^2+76xy-34y^2$	$\dfrac{\sqrt{10400}}{34}$
$(1,89,34)$	$89x^2+199xy-89y^2$	$\dfrac{\sqrt{71285}}{89}$
$(29,169,2)$	$169x^2+367xy-181y^2$	$\dfrac{\sqrt{257045}}{169}$
$(13,194,5)$	$194x^2+432xy-196y^2$	$\dfrac{\sqrt{338720}}{194}$
$(1,233,89)$	$233x^2+521xy-233y^2$	$\dfrac{\sqrt{488597}}{233}$

We use the second observation here to create the relations

$$\mathcal{L}(a,Q,b) = (a, 3aQ-b, Q),$$
$$\mathcal{R}(a,Q,b) = (Q, 3Qb-a, b).$$

We use these relations to build a tree containing regular Markov numbers in the following definition.

Definition 1.1.11. We construct a binary directed tree $\mathcal{G}_\Sigma(1,5,2)$ with root $(1,5,2)$, where each vertex V is connected to the vertices $\mathcal{L}(V)$ and $\mathcal{R}(V)$ by the edges $\big(V,\mathcal{L}(V)\big)$ and $\big(V,\mathcal{R}(V)\big)$ respectively. We call $\mathcal{G}_\Sigma(1,5,2)$ the *tree of regular Markov numbers.*

Remark 1.1.12. A proof that $\mathcal{G}_\Sigma(1,5,2)$ is in fact a tree may be found in the book [2, p.46-47] by M. Aigner.

Example 2. In Figure 1.1 we see the first 4 levels of the tree of regular Markov numbers.

We call a Markov triple *non-singular* if all three elements are different, and *singular* otherwise. The following theorem shows the importance of the tree of regular Markov numbers in terms of finding Markov triples. The proof may be found in [23, 2].

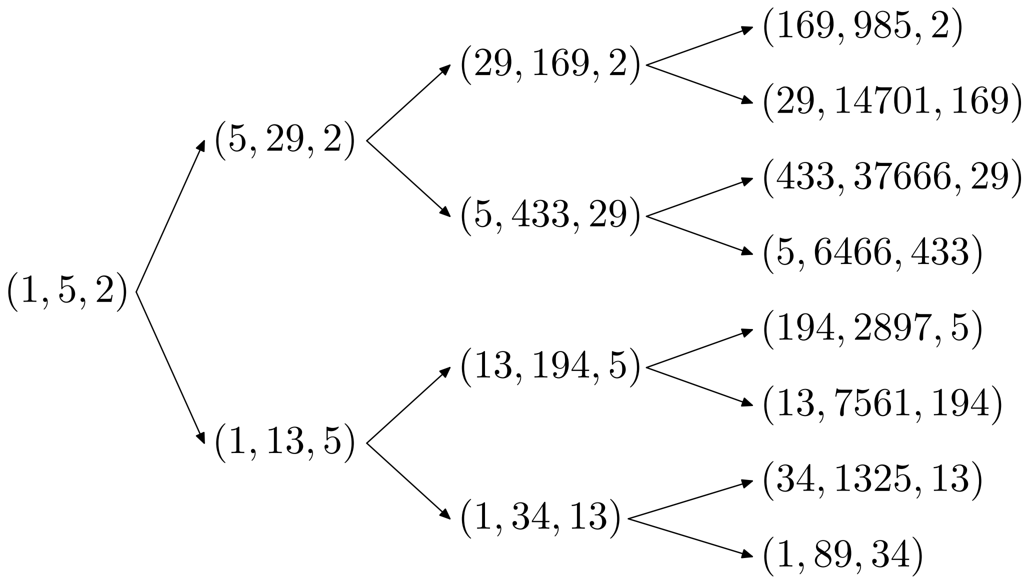

Figure 1.1: The first 4 levels in the tree of regular Markov numbers.

Theorem 1.1.13.

(*i*) *The only singular solutions* (*up to permutation*) *of the Markov Diophantine equation are* $(1, 1, 1)$ *and* $(1, 2, 1)$.

(*ii*) *Every non-singular regular Markov triple appears exactly once in the tree of regular Markov numbers.*

(*iii*) *The triple at any vertex of the tree of regular Markov numbers is a Markov triple.*

Uniqueness conjecture

A famous conjecture first stated by G. Frobenius [38], involving the tree of regular Markov numbers, is the following.

Conjecture 1.1.14 (Regular uniqueness conjecture)**.** Markov triples are uniquely determined by their largest element.

For any two Markov triples (a_1, Q, b_1) and (a_2, M, b_2) with $a_1 \le b_1 \le Q$ and $a_2 \le b_2 \le M$, the conjecture says that if $Q = M$ then $a_1 = a_2$ and $b_1 = b_2$. Many different forms of this conjecture are known, and are the subject of the book by M. Aigner [2].

In Section 1.3 we see a way to define forms associated with certain sequences of positive integers. We extend the theory of Markov numbers by constructing new sequences, and defining forms from them. In Theorem 4.1.14 we show certain conditions on sequences whose associated forms attain their Markov minima at the point $(1, 0)$. From these forms we then define analogues of regular Markov numbers, called *general Markov numbers*, for which we have the following analogue of the uniqueness conjecture.

Conjecture 1.1.15. Triples of general Markov numbers are uniquely defined by their largest element.

We show counterexamples to some cases of this conjecture in Theorem 5.2.8.

1.1.3 Recent work

In this short subsection we mention some of the work that has been completed in the area of Markov numbers since 1950. The uniqueness conjecture is much studied. The excellent book by M. Aigner [2] provides an overview of the topic, including the many areas in mathematics in which the conjecture appears.

There have been a number of attempted proofs, all of which appear to be incorrect, for example [81].

Some cases are known, relating to regular Markov numbers of prime powers. For example, it is known that a Markov triple (a, Q, b) is uniquely defined by Q if either Q, $3Q-2$, or $3Q+2$ is prime, twice a prime, or four times a prime. This was shown by A. Baragar in [5]. A proof for when Q is prime is included in the paper of J. O. Button [14]. M. L. Lang and S. P. Tan [61] showed that the conjecture holds whenever Q is a prime power. Y. Zhang [98] improves the result of A. Baragar [5] to show that the conjecture holds whenever $3Q-2$ or $3Q+2$ is a prime power, twice a prime power, or four times a prime power. Equivalent conditions of the conjecture in terms of Legendre symbols are presented by A. Srinivasan in [90].

Numerical evidence for the conjecture may be found in the paper by D. Rosen and G. S. Patterson [80], who found no counterexamples after computing all Markov numbers with fewer than 31 digits.

Other representations of the uniqueness conjecture have been studied. For example, Y. Bugeaud, C. Reutenauer, and S. Siksek [13] study the conjecture along with Sturmian sequences, certain infinite sequences of positive integers. Equivalent conditions for the conjecture in terms of principal ideals were introduced by J. O. Button in [15]. Principal ideals have also been studied in relation to Markov numbers by A. Srinivasan in [91]. A. N. Rudakov [82] studied the relation of regular Markov numbers to exceptional bundles on the projective plane. This list of recent work in the study of Markov numbers is by no means exhaustive.

In this thesis we study Markov numbers in relation to integer geometry. We recommend the book by O. Karpenkov [52] on integer geometry for interested readers. Aside from integer geometry, Markov numbers have also been studied in relation to hyperbolic geometry. Starting with H. Cohn [20, 21], works on the topic include A. Haas [43], C. Series [84, 85], and more recently B. Springborn [89].

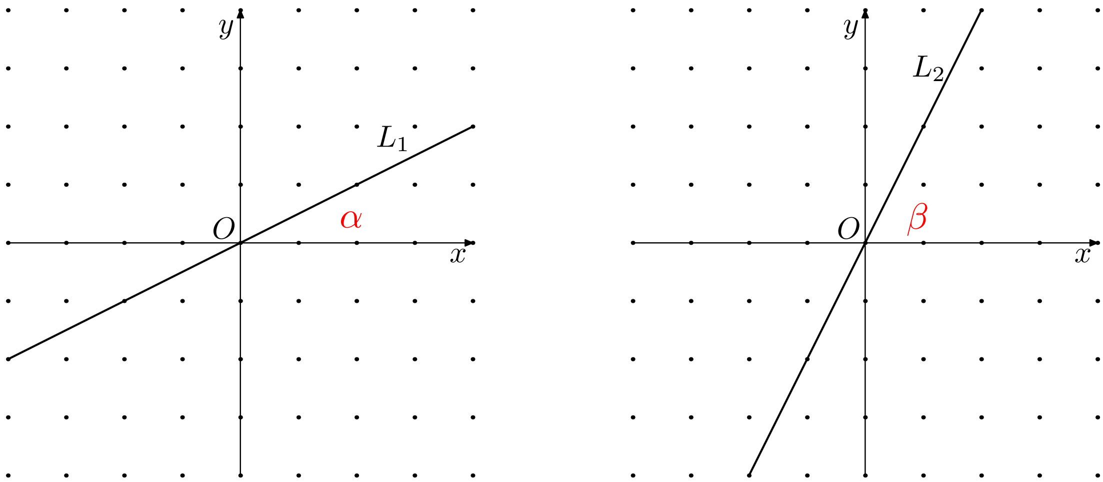

Figure 1.2: Two lines L_1 and L_2.

1.2 Motivating example for integer geometry

In this section we discuss an example that motivates our study of integer geometry in relation to Markov numbers.

Let a_1, b_1, a_2, and b_2 be non zero integers with $\gcd(a_i, b_i) = 1$ for $i = 1, 2$. Let L_1 and L_2 be the lines defined by

$$y = \frac{b_1}{a_1}x \qquad \text{and} \qquad y = \frac{b_2}{a_2}x$$

respectively, as in Figure 1.2. A simple question to ask is: can we transform L_1 to L_2 while preserving the integer lattice?

The answer is yes. Let (u_i, v_i) be pairs of integers such that

$$a_i u_i + b_i v_i = 1$$

(we know (u_i, v_i) exist since $\gcd(a_i, b_i) = 1$). Then let M_1 and M_2 be the matrices

$$M_1 = \begin{pmatrix} u_1 & v_1 \\ -b_1 & a_1 \end{pmatrix} \qquad \text{and} \qquad M_2 = \begin{pmatrix} u_2 & v_2 \\ -b_2 & a_2 \end{pmatrix}.$$

Both M_1 and M_2 are in $\mathrm{SL}(2, \mathbb{Z})$, the group of 2×2 matrices with integer elements and whose determinants are 1. These are matrices that preserve the integer lattice.

The matrix M_1 takes L_1 to the line $y = 0$, and M_2 takes L_2 to the line $y = 0$. Hence any two lines passing through the origin and another integer point are related by an $\mathrm{SL}(2, \mathbb{Z})$ transformation.

Now consider a *two-dimensional arrangement* (or simply, arrangement), a pair of lines intersecting at the origin, see for example Figure 1.3. We ask the question: when are two arrangements $\mathrm{SL}(2, \mathbb{Z})$ equivalent?

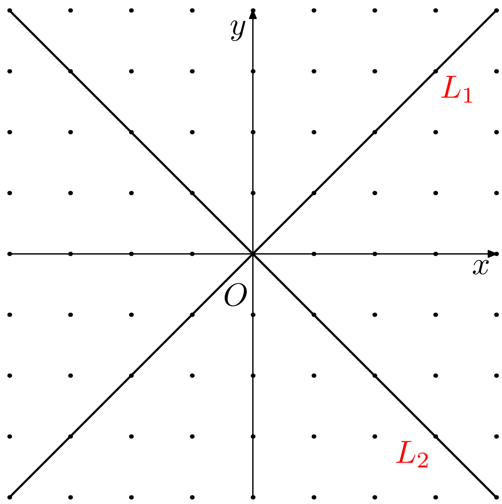

Figure 1.3: The arrangement defined by the lines L_1 and L_2.

This is a more difficult question to answer, but may be shown through integer geometry, see for example the book by O. Karpenkov [52, Chapter 7].

These arrangements are solutions to equations $f(x, y) = 0$, where f is a binary quadratic form

$$f(x, y) = ax^2 + bxy + cy^2$$

with real coefficients a, b, and c, and positive discriminant $b^2 - 4ac$. One may ask at which integer points a form f attains its minimal positive value, and alternatively its maximal negative value. The complete answer to this question is still not known. However, great progress towards this problem was made by A. Markov in his theorem, seen above in Theorem 1.1.9.

In this thesis we extend upon this result using the theory of geometric continued fractions as our main tool. We extend the infinite periodic sequences that are central to the study of regular Markov numbers. These periods are called *Markov sequences*. The periods are related to *Cohn words*, which are also known as *Christoffel words*.

1.3 Background of integer geometry

The geometry of numbers plays a large part in this thesis, specifically integer geometry. A good introduction to the topic is the book by J. W. S. Cassels [18]. More specific to our work is the book by O. Karpenkov [52], based on his earlier works [49, 50, 51]. These in turn have their basis in the work of F. Klein [58], who introduced a notion of the geometrisation of continued fractions.

Although not considered here, it is worth noting that extending the notion of geometrical continued fractions to higher dimensions has been studied. We make particular reference to the book by F. Schweiger [83] and the paper of O. Karpenkov and A. Ustinov [53].

In this section we give an overview of the notions and definitions that form the basis of the study of integer geometry. We start in Subsection 1.3.1, where we recall some classical results for regular continued fractions. The concepts in integer geometry important to our work are covered in Subsection 1.3.2.

We introduce some results on the relation of integer geometry and the values of forms in Subsection 1.3.3.

1.3.1 Continued fractions

Continued fractions are an important tool in the study of Markov numbers. Good introductions to regular continued fractions are the books by J. W. S. Cassels [17] and A. Khinchin [57], and the excellent book by O. Perron [75]. Various other interesting works on continued fractions include the papers by M. Hall [44] dealing with sums and products of continued fractions, E. M. Wright's paper [95] on Diophantine approximation, a paper by O. Karpenkov [51] on the second Kepler law, and a paper on factorising integers by R. A. Mollin. [67]. One may relate continued fractions with triangulations as seen in the paper of S. Morier-Genoud and V. Ovsienko [71].

Let us define regular continued fractions.

Definition 1.3.1.

(i) Let $(a_i)_0^n$ be a sequence of real numbers. The expression

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

is called the *continued fraction* for the sequence $(a_i)_0^n$. We denote the regular continued fraction by $[a_0; a_1 : a_2 : \ldots : a_n]$.

(*ii*) For an infinite sequence of real numbers $(a_i)_0^\infty$, if the limit

$$\lim_{k\to\infty}[a_0; \ldots : a_k]$$

exists then we call it the *infinite continued fraction* of $(a_i)_0^\infty$.

(*iii*) For an eventually periodic sequence of real numbers $b_0, \ldots, b_m, a_0, \ldots$ with period $(a_0, \ldots, a_n)$ we denote its infinite continued fraction by

$$[b_0; \ldots : b_m : \langle a_0, \ldots, a_n\rangle],$$

where the sequence inside the brackets $\langle\rangle$ is the period.

(*iv*) If a_0 is an integer and $(a_i)_1^n$ is a sequence of positive integers then

$$[a_0; a_1 : \ldots : a_n]$$

is called the *regular continued fraction* of $(a_i)_0^n$.

We present some well known results on continued fractions. The proofs may be found in the books of A. Khinchin [57], J. W. S. Cassels [17], and O. Perron [75].

Proposition 1.3.2.

(*i*) *For a rational number α there are exactly two regular continued fraction expansions, one with an even number of elements and one with an odd number of elements.*

$$\alpha = [a_0; a_1 : \ldots : a_n] = [a_0; a_1 : \ldots : a_n - 1 : 1],$$

where $a_n > 1$.

(*ii*) *For any irrational number a continued fraction expansion exists, and is unique.*

(*iii*) *There is a 1-1 correspondence between periodic continued fractions and roots of quadratic polynomials.*

The result of Proposition 1.3.2 (*iii*) goes back to L. Euler and J. L. Lagrange. L. Euler proved that any periodic regular continued fraction is a root of a quadratic polynomial in the paper [36], for which there is an English translation here [96]. J. L. Lagrange proved the converse of this statement in [60].

Example 3. The even and odd continued fraction expansions of 7/3 are

$$\frac{7}{3} = 2 + \frac{1}{3} = 2 + \cfrac{1}{2 + \cfrac{1}{1}}.$$

Example 4. The continued fraction expansion of π starts

$$\pi = [3; 7 : 15 : 1 : 292 : 1 : 1 : 1 : 2 : 1 : 3 : \ldots].$$

The positive root of $x^2 - 2$ is $\sqrt{2}$, whose periodic continued fraction is

$$\sqrt{2} = [1; \langle 2\rangle].$$

1.3.2 Concepts of integer geometry

Integer geometry in two dimensions is the study of objects invariant under any transformation that preserves the integer lattice $\mathbb{Z}^2$. We define the basic notions in this section. We give further technical details in Chapter 2. The definitions in this subsection are taken from our general reference on the subject, the book by O. Karpenkov [52]. Resources on the topic of the geometry of continued fractions include the book by F. Schweiger [83] and the paper of O. Karpenkov [53].

Recall that the number of left cosets of a subgroup H in a group G is called the *index* of H in G.

Definition 1.3.3. Let us define the objects being considered in integer geometry.

(i) The pair (a, b) is an *integer point* if both a and b are integers.

(ii) An *angle* $\angle AOB$ is an ordered pair of rays sharing the same endpoint, or vertex, O. The points A and B are contained on separate rays.

(iii) A *rational angle* is an angle whose vertex is an integer point, and whose edges both contain integer points other than the vertex. If an angle has an integer point as its vertex, but is not a rational angle, it is called an *irrational angle*.

(iv) An *integer segment* is a line segment whose endpoints are integer points.

(v) An *integer triangle* is a triangle $\triangle ABC$ whose vertices are all integer points.

(vi) An *integer polygon* is a polygon $A_1A_2\ldots A_n$ whose vertices are all integer points.

Remark 1.3.4. Note that $\angle AOB$ and $\angle BOA$ are different angles.

Recall that $\mathrm{Aff}(2, \mathbb{Z})$ is the semi-direct product of the group of lattice preserving translations and $\mathrm{GL}(2, \mathbb{Z})$, 2×2 integer matrices with determinant equal to ±1.

Definition 1.3.5. Two objects defined in Definition 1.3.3 are *integer congruent* if they are equivalent up to an affine transformation of the integer lattice.

We introduce the integer area for integer triangles.

Definition (Integer area lS). The *integer area of an integer triangle* $\mathrm{lS}(\triangle ABC)$ is the index of the sublattice generated by the vectors AB and BC in the integer lattice.

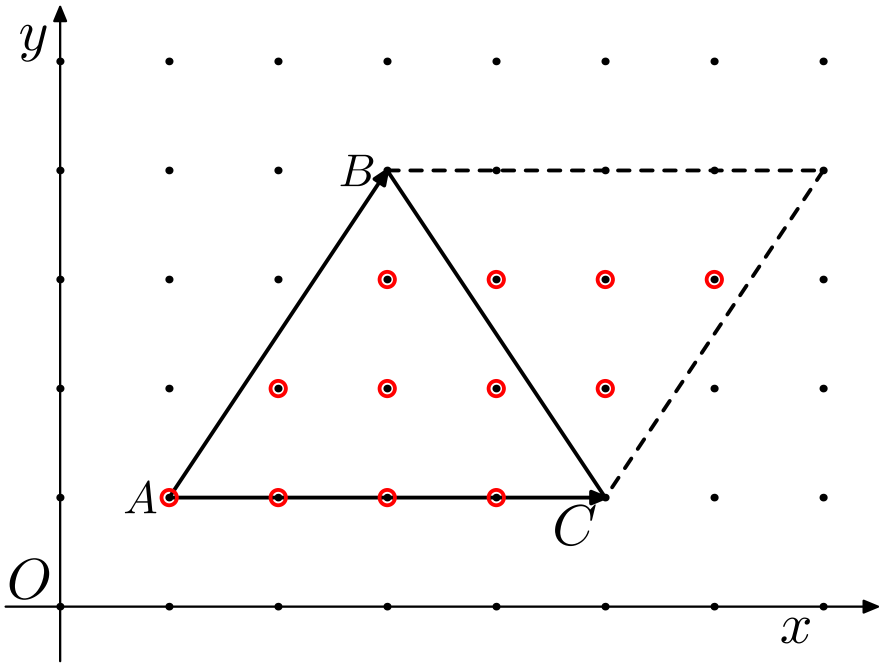

Figure 1.4: The point counting for the integer area of the triangle $\triangle ABC$.

Remark 1.3.6. In this thesis we read the notation lS as *integer area*. In the classical theory of integer geometry this concept is sometimes called the *lattice area*. It is sometimes denoted IS, but we use the notation lS throughout.

The integer area of an integer triangle may be found by counting the points in the parallelogram defined by any two sides of that triangle. This fact is proven in [52, Prop. 2.2].

Specifically, let $\triangle ABC$ be an integer triangle. Let $\overrightarrow{v} = \overrightarrow{AB}$ and $\overrightarrow{w} = \overrightarrow{AC}$, and let S be the set defined by

$$S = \{a\overrightarrow{v} + b\overrightarrow{w} \mid 0 \leq a < 1, 0 \leq b < 1\}.$$

Then $\mathrm{lS}(\triangle ABC)$ is the number of integer points in S.

Example 5. Figure 1.4 shows the integer triangle with vertices

$$A = (1, 1), \quad B = (3, 4), \quad C = (5, 1).$$

The integer area is $\mathrm{lS}(\triangle ABC) = 12$.

Remark 1.3.7. Let $\triangle ABC$ be an integer triangle with coordinates $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$. Let the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ have coordinates

$$\overrightarrow{AB} = (a_1 - b_1, a_2 - b_2), \quad \overrightarrow{AC} = (a_1 - c_1, a_2 - c_2).$$

Recall that the Euclidean area of the $\triangle ABC$ is

$$\frac{1}{2}\left|\det\left(\overrightarrow{AB}, \overrightarrow{AC}\right)\right|.$$

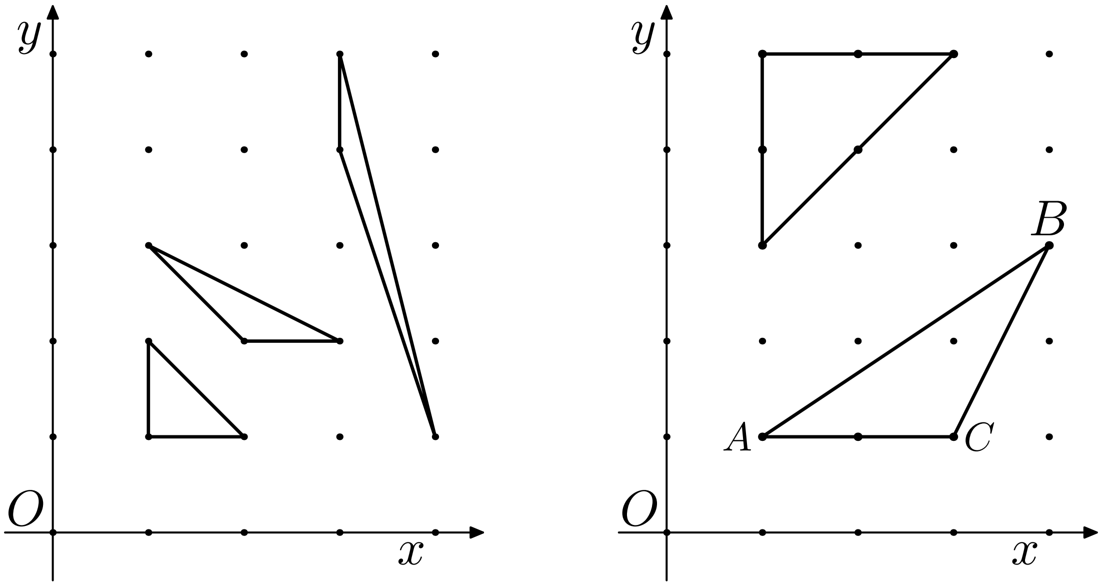

Figure 1.5: **Left hand side:** some triangles with lS = 1. **Right hand side:** some triangles with lS = 4.

From [52, Cor. 2.15] we have that the integer area of $\triangle ABC$ is twice its Euclidean area, so

$$\mathrm{lS}(\triangle ABC) = \left|\det\left(\overrightarrow{AB}, \overrightarrow{AC}\right)\right|.$$

Example 6. The integer area of the triangle in Figure 1.4 is

$$\mathrm{lS}(\triangle ABC) = \left|\det\begin{pmatrix} -2 & -3 \\ -4 & 0 \end{pmatrix}\right| = 12.$$

Example 7. Figure 1.5 shows some integer triangles and their integer area. For the triangle with vertices

$$A = (1,1), \quad B = (4,3), \quad C = (3,1),$$

we have

$$\mathrm{lS}(\triangle ABC) = \left|\det\begin{pmatrix} -3 & -2 \\ -2 & 0 \end{pmatrix}\right| = 4.$$

Definition (**Integer length** $\mathrm{l}\ell$)**.** Let A and B be integer points. Define the *integer length* of the line segment AB to be the number of integer points in the line segment minus 1. We denote this by $\mathrm{l}\ell(AB)$.

Remark 1.3.8. From [52, Def. 2.4] we have that the integer length of the line segment AB is the index of the sublattice generated by the vector $\overrightarrow{AB}$ in the lattice of integer points in the line containing AB.

Remark 1.3.9. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be integer points. Note that the integer length of the line segment AB is given by

$$\mathrm{l}\ell(AB) = \gcd(y_2 - y_1, x_2 - x_1).$$

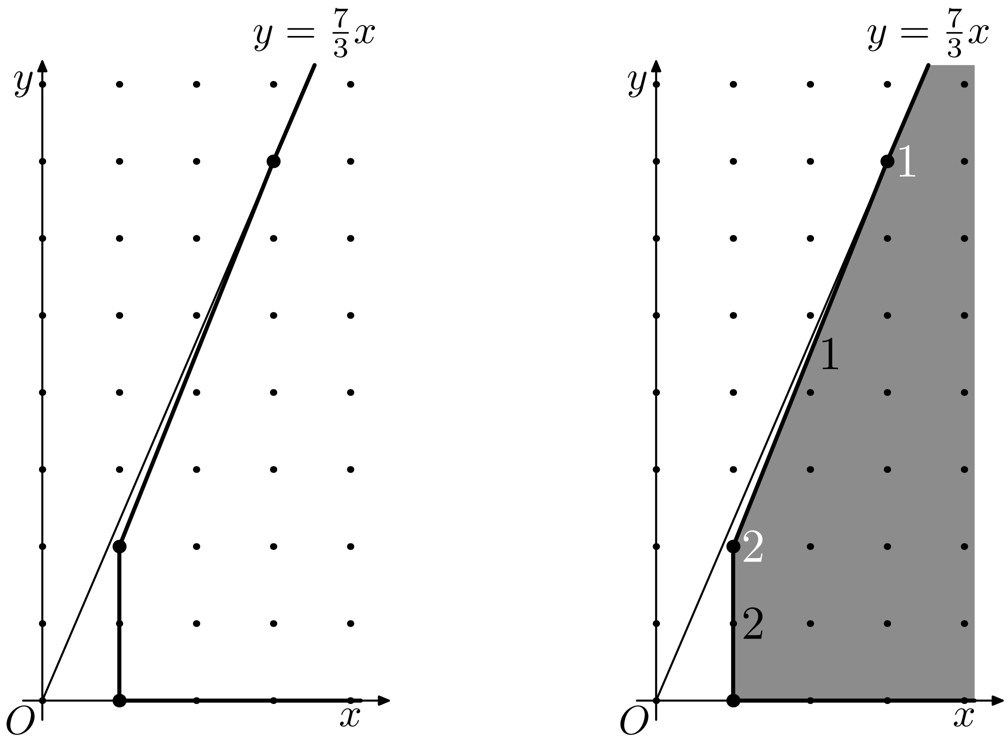

Figure 1.6: A sail and an LLS sequence for the form $7xy - 3y^2$.

Definition 1.3.10 (Orientation). Let the points $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ be in $\mathbb{R}^2$. Denote the interior angle made by the intersection of line segments AB and BC by $\angle ABC$. Define the *oriented area* of the parallelogram given by sides BA and BC as

$$\mathrm{lS}^*(ABC) = \det \begin{pmatrix} b_1 - a_1 & b_2 - a_2 \\ b_1 - c_1 & b_2 - c_2 \end{pmatrix}.$$

Define the *orientation* of the angle $\angle ABC$ as

$$\operatorname{sign}\big(\mathrm{lS}^*(ABC)\big).$$

Remark 1.3.11. Note that the angles $\angle ABC$ and $\angle CBA$ have different orientations. Note also that the orientation of $\angle ABC$ is given explicitly by

$$\operatorname{sign}\big(\mathrm{lS}^*(ABC)\big) = \operatorname{sign}\big((b_1 - a_1)(b_2 - c_2) - (b_2 - a_2)(b_1 - c_1)\big).$$

Definition 1.3.12. Consider α, one of the four angles formed by two lines whose intersection is an integer point A. The smallest convex set (known as the *convex hull*) containing all integer points excluding A inside α has a boundary that is a broken line. We call this broken line the *sail of* α. Since the sail is convex it may be defined by the set of vertices of the broken line.

Example 8. The left hand side of Figure 1.6 shows the sail for the angle containing the point $(1, 1)$ given by the form $7xy - 3y^2$. Figure 1.7 shows the infinite sails of the form $5x^2 + 14xy - 10y^2$.

Now we define an important notion in integer geometry, the LLS sequence.

Definition 1.3.13.

(*i*) Let α be the rational angle $\angle ABC$ where A and B are integer points. The *integer sine of* α is

$$\mathrm{lsin}(\alpha) = \frac{\mathrm{lS}(\triangle ABC)}{\mathrm{l}\ell(AB)\,\mathrm{l}\ell(BC)}.$$

(*ii*) Let $\alpha = \angle BOC$ be an angle whose vertex is the origin $O = (0,0)$. Denote by (A_i) the sequence of vertices for the sail of α such that the vertices $(A_i)_{i<0}$ approach the ray containing the point B and the vertices $(A_i)_{i>0}$ approach the ray containing the point C. Define

$$a_{2k} = \mathrm{l}\ell(A_k A_{k+1}),$$
$$a_{2k-1} = \mathrm{lsin}(A_{k-1}A_k A_{k+1}),$$

for all admissible k. The sequence (a_i) is called the *lattice length sine sequence* (LLS sequence) of the angle α.

Remark 1.3.14. The Euclidean area of a triangle $\triangle ABC$, which we denote here by $\mathrm{S}(\triangle ABC)$, may be written

$$\mathrm{S}(\triangle ABC) = \frac{AC \sin(\angle ABC)}{2}.$$

Rearranging we have

$$\sin(\angle ABC) = \frac{2\,\mathrm{S}(\triangle ABC)}{AC}.$$

This equation motivates our definition of integer sine, with the Euclidean area $\mathrm{S}(\triangle ABC)$ replaced with the integer area $\mathrm{lS}(\triangle ABC)$ and the lengths A and C replaced with the integer lengths $\mathrm{l}\ell(AB)$ and $\mathrm{l}\ell(BC)$.

Example 9. The right hand side of Figure 1.6 shows the LLS sequence for the sail of the angle containing the point $(1,1)$ given by the form $7xy - 3y^2$. The lengths are given in black, and the integer sines are written in white.

The following proposition is proven in [52].

Proposition 1.3.15. *Every sail has an LLS sequence, and every LLS sequence describes a sail.*

The LLS sequence is important since it is an invariant of angles under lattice preserving transformations, as the following theorem proven by O. Karpenkov in [52, Cor. 4.9, Thm 4.10] shows.

Theorem 1.3.16.

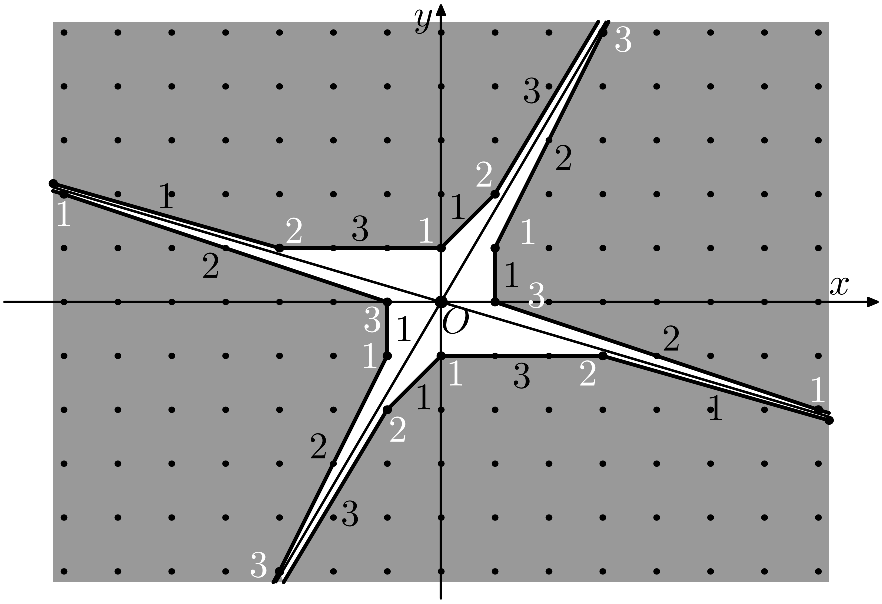

Figure 1.7: The LLS sequences of the sails for the form $f(x,y) = 5x^2+14xy-10y^2$. The integer lengths are written in black, the integer sines in white.

(*i*) *LLS sequences are invariant under* $\mathrm{Aff}(2,\mathbb{Z})$ *transformations.*

(*ii*) *Two angles with an integer point as their vertices are integer congruent if and only if they have the same LLS sequence.*

Definition 1.3.17. If the LLS sequence of one sail is the reverse (up to a shift of index) of the LLS sequence of another sail, then the two sails are said to be *dual.*

The following proposition is proven in [52].

Proposition 1.3.18. *The solutions to* $f(x,y) = 0$ *for a form* f *split the plane into 4 angles. The sails of opposite angles are congruent. The sails of adjacent angles are dual.*

Example 10. Figure 1.7 shows the congruency of opposite sails and duality of adjacent sails for the form $5x^2 + 14xy - 10y^2$.

1.3.3 Perron identity

We give a short note on a theorem relating the LLS sequence of a form and its Markov value. The theorem goes back to O. Perron [76].

Theorem 1.3.19 (O. Perron)**.** *Let* (a_i) *be the LLS sequence of a form* f*. Then the Markov value of* f *is given by the following relation* (*known as the* Perron identity)

$$\frac{\sqrt{\Delta(f)}}{m(f)} = \sup_{i\in\mathbb{Z}} \Big\{ a_i + [0; a_{i+1} : a_{i+2} : \ldots] + [0; a_{i-1} : a_{i-2} : \ldots] \Big\}.$$

Table 1.2: LLS sequences and Markov values

α	$\sqrt{\Delta(f_\alpha)}/m(f_\alpha)$
$(2,2,1,1,2,2,1,1,2,2)$	$\frac{\sqrt{1687397}}{463}$
$(2,1,1,2,2,1,1,2,2,2)$	$\frac{\sqrt{1687397}}{433}$
$(1,1,2,2,1,1,2,2,2,2)$	$\frac{\sqrt{1687397}}{613}$
$(1,2,2,1,1,2,2,2,2,1)$	$\frac{\sqrt{1687397}}{611}$
$(2,2,1,1,2,2,2,2,1,1)$	$\frac{\sqrt{1687397}}{437}$

Example 11. Let f be the form

$$f(x,y) = 433x^2 + 791xy - 613y^2.$$

From Markov's theorem, Theorem 1.1.9, we have that the Markov value of f is

$$\frac{\sqrt{1687397}}{433}.$$

The LLS sequence for this form is $(1,1,2,2,1,1,2,2,2,2)$. Note that the cyclic shift $(2,2,1,1,2,2,1,1,2,2)$ is palindromic, and hence there are only five possible values for

$$a_i + [0; a_{i+1} : a_{i+2} : \ldots] + [0; a_{i-1} : a_{i-2} : \ldots].$$

One of the values is

$$\begin{aligned}
&2 + [0; \langle 2,1,1,2,2,1,1,2,2,2\rangle] + [0; \langle 2,2,1,1,2,2,1,1,2,2\rangle] \\
=&2 + \frac{-941 + \sqrt{1687397}}{926} + \frac{-911 + \sqrt{1687397}}{926} \\
=&\frac{\sqrt{1687397}}{463}.
\end{aligned}$$

The other values are given in Table 1.2. Hence

$$\begin{aligned}
\frac{\sqrt{\Delta(f)}}{m(f)} &= \sup_{i\in\mathbb{Z}} \Big\{ a_i + [0; a_{i+1} : a_{i+2} : \ldots] + [0; a_{i-1} : a_{i-2} : \ldots] \Big\} \\
&= \frac{\sqrt{1687397}}{463}.
\end{aligned}$$

In Section 2.3 we extend the theorem of Perron to finding any non zero value of the form in terms of continued fractions. The statement is proven in Theorem 2.3.14, and is published in the joint paper with O. Karpenkov [54].

For certain sequences we show that condition in Theorem 1.3.19 of finding the supremum of an infinite set may be simplified. The simplification is given in Theorem 4.4.2.

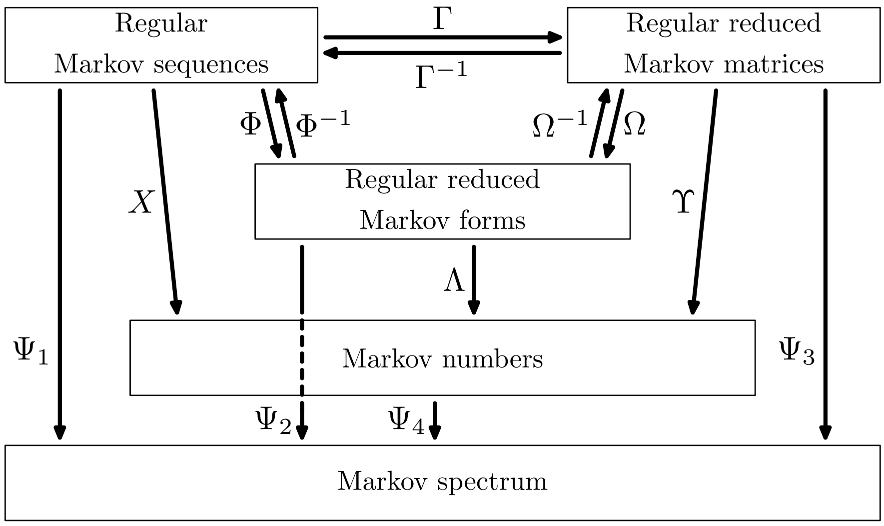

Figure 1.8: The classical structure of Markov theory.

1.4 Research layout

In this thesis we propose an extension to the theory of regular Markov numbers. We do this through the study of the geometry of continued fractions.

In this section we give a brief summary of what is contained in each chapter of this thesis.

Summary of Chapter 2

In Chapter 2 we define the technical notions used throughout the thesis. We study continuants, functions relating to regular continued fractions in Section 2.1. We lay down the important aspects of $\mathrm{SL}(2,\mathbb{Z})$, and go through the proofs of known results in detail in Section 2.2.
In Section 2.3 we link the Markov spectrum to geometry through the Perron identity, specifically by looking at continued fractions associated with broken lines. The key result of this is Theorem 2.3.14, proven in our joint paper with O. Karpenkov [54].
We close the chapter with Section 2.5 where we describe the Markov spectrum in more detail, and introduce Cohn matrices. These matrices play an important role in our work.

Summary of Chapter 3

In Chapter 3 we study the matrices, forms, and sequences related to regular Markov numbers. We begin by defining the triple graph structure that is central

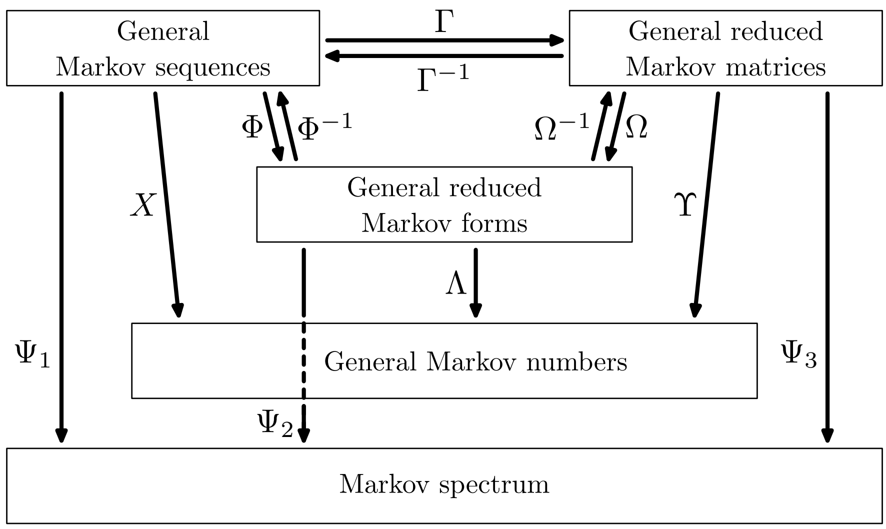

Figure 1.9: The extended structure of Markov theory.

to our study in Section 3.1. We describe regular Markov numbers with this notation.

We define sequences related to regular Markov numbers in Section 3.2, and show they have a triple graph structure. These sequences are related to Christoffel words and Cohn words.

Markov forms and Cohn matrices are representatives of equivalence classes of forms and matrices. These classes are closely related in that they describe the Markov spectrum below three. However, the specific form and matrix for a given regular Markov number describe different, although equivalent, arrangements.

We pick new representatives for matrices and forms related to regular Markov numbers in Section 3.2, ones which describe the same two-dimensional arrangement. We show that both the new matrices and forms have a triple graph structure, which is introduced in Section 3.1.

We show relations between the newly defined graphs of sequences, matrices, and forms, and the tree of regular Markov numbers in Section 3.3.

In Section 3.4 we relate all of the triple graphs to the Markov spectrum below three, culminating in the diagram in Figure 1.8.

Summary of Chapter 4

In Chapter 4 we extend the notion of Markov numbers. In the previous chapter we saw how the graphs of matrices, forms, and regular Markov numbers may be defined directly from a graph of sequences. In Section 4.1 we define different sequences with which to generate graphs, which we call Markov sequences. The key results to defining these Markov sequences are Theorems 4.1.14 and 4.1.27.

We prove Theorem 4.1.27 in Section 4.2.

Markov sequences define new graphs of forms, matrices, and numbers (called general Markov numbers). We define the relations between these graphs in Section 4.3

We finalise the relations between these graphs and the Markov spectrum in Section 4.4. The key result in this section is Proposition 4.4.2, that relates the element on the Markov spectrum for a Markov sequence to a continued fraction expansion of that sequence. This is a specific application of the Perron identity. This work is represented in the diagram of Figure 1.9. The diagram is explained fully in Sections 4.3 and 4.4.

Summary of Chapter 5

In Chapter 5 we study the general Markov numbers defined in the previous chapter. We show in Section 5.1 that there are more complicated recurrence relations for general Markov numbers than for regular Markov numbers, a result in Theorem 5.1.5.

There is an analogue to the uniqueness conjecture for general Markov numbers. In Section 5.2 we show that for certain graphs of general Markov numbers this uniqueness conjecture fails. This is the result of Theorem 5.2.8.

We complete our study in Section 5.3 by mentioning some open questions in the area of Markov numbers that we hope to develop further.

Chapter 2

Background and definitions

This chapter introduces the technical notions and definitions used in this thesis. We include known results that are used throughout the text, and also some basic preliminary results that we prove.

We start in Section 2.1 where we introduce the notion of a continuant, and relate these continuants to regular continued fractions. We define palindromic sequences, a concept important to our study of Markov sequences.

We see the effect that $\mathrm{SL}(2, \mathbb{Z})$ transformations have on points, lines, and forms in Section 2.2.

In Section 2.3 we define LLS sequences and continued fractions for arbitrary broken lines. Using this we build up to a relation between the values of forms at any point P other than the origin, and the value of the continued fraction of certain broken lines containing P as a vertex.

Finally in Section 2.5 we look at the Markov spectrum in more detail. We recall important results on the Markov spectrum below 3, and how this relates to equivalence classes of forms. We introduce Cohn matrices, used by H. Cohn to study the Markov spectrum in [20]. We see the triple graph structure of Cohn matrices, which is integral to our study of forms.

2.1 Continuants

Having defined regular continued fractions in Subsection 1.3.1 we now introduce the concept of continuants. These are related to the numerators and denominators of continued fractions, and are described in good detail in [42, pp.301–309].

Example 12. Let (a_1, a_2, a_3, a_4) be a sequence of positive integers. We have that

$$\begin{aligned}
[a_1] &= \frac{a_1}{1}, \\
[a_1; a_2] &= \frac{a_1a_2 + 1}{a_2}, \\
[a_1; a_2 : a_3] &= \frac{a_1a_2a_3 + a_1 + a_3}{a_2a_3 + 1}, \\
[a_1; a_2 : a_3 : a_4] &= \frac{a_1a_2a_3a_4 + a_1a_4 + a_3a_4 + a_1a_2 + 1}{a_1a_2a_3 + a_1 + a_3}.
\end{aligned}$$

The numerators and denominators of the continued fraction may be written as polynomials of the elements. These polynomials play an important role in our study.

Definition 2.1.1. Let $(a_1, \ldots, a_n)$ be a sequence of positive real numbers. Let

$$[a_1; a_2 : \ldots : a_n] = \frac{P}{Q}$$

for some real numbers P and Q. We call P the *continuant of* $(a_1, \ldots, a_n)$.

From the classical theory of continued fractions there is a recursive relation for continuants.

Definition 2.1.2. Let $x_1, x_2, \ldots$ be variables. We define a polynomial $K_n(x_1, \ldots, x_n)$ recursively by

$$\begin{aligned}
K_0() &= 1, \\
K_1(x_1) &= x_1, \\
K_k(x_1, \ldots, x_k) &= x_1K_{k-1}(x_2, \ldots, x_k) + K_{k-2}(x_3, \ldots, x_k).
\end{aligned}$$

Example 13. Let $\alpha = (1, 1, 2, 2)$. Then

$$\begin{aligned}
K_4(\alpha) &= K_4(1, 1, 2, 2) \\
&= 1 \cdot K_3(1, 2, 2) + K_2(2, 2) \\
&= 1 \cdot \big(1 \cdot K_2(2, 2) + K_1(2)\big) + 2 \cdot K_1(2) + K_0() \\
&= 2 \cdot K_1(2) + K() + K_1(2) + 2 \cdot K_1(2) + K_0() \\
&= 4 + 1 + 2 + 4 + 1 \\
&= 12.
\end{aligned}$$

Note that the continued fraction of α is

$$[1;1:2:2] = \frac{12}{7} = \frac{K_4(1,1,2,2)}{K_3(1,2,2)}.$$

The following proposition relates $K_n(a_1,\ldots,a_n)$ to the continuant of $(a_1,\ldots,a_n)$.

Proposition 2.1.3. *The continuant of $(a_1,\ldots,a_n)$ is $K_n(a_1,\ldots,a_n)$. Moreover, the continued fraction of $(a_1,\ldots,a_n)$ is given in terms of continuants*

$$[a_1;a_2:\ldots:a_n] = \frac{K_n(a_1,\ldots,a_n)}{K_{n-1}(a_2,\ldots,a_n)}.$$

Remark 2.1.4. From here on we use the term *continuant of* $\alpha = (a_i)_1^n$ to mean $K_n(\alpha)$.

We use a slightly different notation for continuants to simplify notation in long calculations.

Definition 2.1.5. For a sequence $\alpha = (a_1,\ldots,a_n)$ we write

$$K_i^j(\alpha) = \begin{cases} K_{j-i+1}(a_i,\ldots,a_j) & \text{if } 1 \le i \le j \le n, \\ 1 & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

We will often see the continuant $K_1^{n-1}(\alpha)$, and so shorten it with the following notation

$$\breve{K}(\alpha) = K_1^{n-1}(\alpha).$$

Example 14. Let $\alpha = (1,1,2,2)$. Then

$$\begin{aligned} K_2^4(\alpha) &= K_2^4(1,1,2,2) \\ &= 1\cdot K_3^4(\alpha) + K_4^4(\alpha) \\ &= 2\cdot K_4^4(\alpha) + K_5^4(\alpha) + K_4^4(\alpha) \\ &= 3\cdot K_4^4(\alpha) + 1 \\ &= 7 \end{aligned} \qquad \begin{aligned} K_2^4(\alpha) &= K_3(1,2,2) \\ &= 1\cdot K_2(2,2) + K_1(2) \\ &= 2\cdot K_1(2) + K_0() + K_1(2) \\ &= 3\cdot K_1(2) + K_0() \\ &= 7. \end{aligned}$$

The following proposition on splitting continuants is used throughout this work. One may find a proof in [42].

Proposition 2.1.6. *Let $(a_1,\ldots,a_n)$ be an arbitrary sequence and $1 \le k < n$. Then we have*

$$K_n(\alpha) = K_1^k(\alpha)K_{k+1}^n(\alpha) + K_1^{k-1}(\alpha)K_{k+2}^n(\alpha).$$

Example 15. Let $\alpha = (1,1,2,2,1,1,2,2)$. We have

$$K_1^4(\alpha) = 12, \qquad K_1^3(\alpha) = K_3(1,1,2) = 5, \qquad K_3(1,2,2) = 7.$$

We split the continuant $K_1^8(\alpha)$ according to the formula of Proposition 2.1.6 in the following way, adding the product of the under brace continuants to the product of the over brace continuants.

$$\Big(\underbrace{\overbrace{1,1,2}^{K_1^3(\alpha)},2}_{K_1^4(\alpha)},\underbrace{1,\overbrace{1,2,2}^{K_5^8(\alpha)}}_{K_5^8(\alpha)}\Big).$$

Hence

$$K_1^8(\alpha) = K_1^4(\alpha)\cdot K_5^8(\alpha) + K_1^3(\alpha)\cdot K_6^8(\alpha) = 144 + 35 = 179.$$

Definition 2.1.7.

(i) Let $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_m)$ be finite sequences. Define the *concatenation* of α and β to be

$$\alpha \oplus \beta = (a_1, \ldots, a_n, b_1, \ldots, b_m).$$

We often shorten $\alpha \oplus \beta$ to just $\alpha\beta$.

(ii) A sequence $\alpha = (a_1, \ldots, a_n)$ is called *palindromic* if

$$(a_1, \ldots, a_n) = (a_n, \ldots, a_1).$$

We write $\overline{\alpha}$ to denote the reverse of α, so

$$\overline{\alpha} = (a_n, \ldots, a_1).$$

We present some classical results on continuants. The proofs may be found for example in the books by R. L. Graham, D. E. Knuth, and O. Patashnik [42] or O. Karpenkov [52].

Proposition 2.1.8. *Let $\alpha = (a_1, \ldots, a_n)$ be a sequence of positive integers.*

(i) *The* gcd *of $K_1^n(\alpha)$ and $K_2^n(\alpha)$ is*

$$\gcd\big(K_1^n(\alpha), K_2^n(\alpha)\big) = 1.$$

(ii) *Continuants satisfy*

$$K_1^n(\alpha) = K_1^n\left(\overline{\alpha}\right).$$

We call this the symmetric property of continuants.

(iii) *We have that*

$$K_1^n(\alpha)K_2^{n-1}(\alpha) - K_2^n(\alpha)K_1^{n-1}(\alpha) = (-1)^n.$$

(iv) *If $a_n > 1$ then*

$$K_1^n(a_1, \ldots, a_{n-1}, a_n) = K_1^{n+1}(a_1, \ldots, a_{n-1}, a_n - 1, 1).$$

2.2 A note on $\mathrm{SL}(2,\mathbb{Z})$ transformations

In this section we discuss the effect of $\mathrm{SL}(2,\mathbb{Z})$ transformations on certain geometric objects. The proofs in this section are classical, and may be found in the book [52]. We work through the proofs in greater detail here, to illustrate clearly what is happening.
In Subsection 2.2.1 we go through the important invariants of $\mathrm{SL}(2,\mathbb{Z})$ transformations, in particular convexity.
In Subsection 2.2.2 we show how $\mathrm{SL}(2,\mathbb{Z})$ affects forms and their sails, with particular attention to the values forms attain on their sails.

2.2.1 $\mathrm{SL}(2,\mathbb{Z})$ invariants

Recall the definition of orientation from Definition 1.3.10.

Proposition 2.2.1. $\mathrm{SL}(2,\mathbb{Z})$ *preserves orientation.*

Proof. Let $X = (x_1, x_2)$, $Y = (y_1, y_2)$, and $Z = (z_1, z_2)$ be points in $\mathbb{R}^2$. The angle $\angle XYZ$ has orientation

$$\operatorname{sign}\big((y_1 - x_1)(y_2 - z_2) - (y_2 - x_2)(y_1 - z_1)\big).$$

Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an $\mathrm{SL}(2,\mathbb{Z})$ transformation. The points X, Y, and Z are mapped by M to

$$\begin{aligned} MX &= (ax_1 + bx_2, cx_1 + dx_2), \\ MY &= (ay_1 + by_2, cy_1 + dy_2), \\ MZ &= (az_1 + bz_2, cz_1 + dz_2). \end{aligned}$$

Define T by

$$\begin{aligned} T = \Big(&\big(ay_1 + by_2 - ax_1 - bx_2\big)\big(cy_1 + dy_2 - cz_1 - dz_2\big) \\ &-\big(cy_1 + dy_2 - cx_1 - dx_2\big)\big(ay_1 + by_2 - az_1 - bz_2\big)\Big). \end{aligned}$$

Then the orientation of the angle $\angle MXMYMZ$ is $\operatorname{sign}(T)$. We have that

$$\begin{aligned} T =& acy_1^2 - acy_1^2 + bdy_2^2 - bdy_2^2 + acx_1z_1 - acx_1z_1 + bdx_2z_2 - bdx_2z_2 \\ &+ ady_1y_2 - bcy_1y_2 - ady_1z_2 + bcy_1z_2 + adx_1z_2 - bcx_1z_2 - adx_1y_2 + bcx_1y_2 \\ &- bdy_2z_2 + bdy_2z_2 - bdx_2y_2 + bdx_2y_2 - acy_1z_1 + acy_1z_1 - acx_1y_1 + acx_1y_1 \\ &+ bcy_1y_2 - ady_1y_2 + ady_2z_1 - bcy_2z_1 + adx_2y_1 - bcx_2y_1 + bcx_2z_1 - adx_2z_1, \end{aligned}$$

and so

$$\begin{aligned} T &= y_1y_2 - y_1z_2 + x_1z_2 - x_1y_2 - y_1y_2 + y_2z_1 + x_2y_1 - x_2z_1 \\ &= (y_1 - x_1)(y_2 - z_2) - (y_2 - x_2)(y_1 - z_1). \end{aligned}$$

This completes the proof. ■

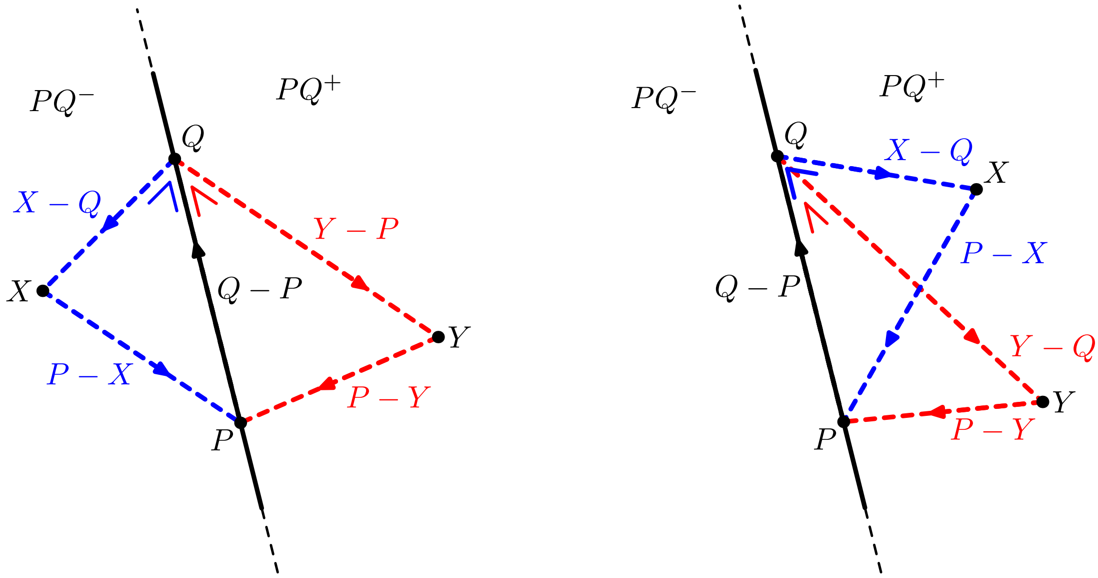

Figure 2.1: How orientation shows whether points are on the same half plane or not.

Definition 2.2.2. Let P and Q be two points in $\mathbb{R}^2$. The line containing P and Q in $\mathbb{R}^2$ splits the plane into two halfplanes. Let $\hat{r}_1$ be the unit vector contained in the line PQ, based in P and in the direction of Q. Let

$$\hat{r}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \hat{r}_1,$$

Both $\hat{r}_1$ and $\hat{r}_2$ are unit vectors and form a basis in $\mathbb{R}^2$. We denote the halfplane containing $\hat{r}_2$ by PQ^+, and the halfplane containing $-\hat{r}_2$ by PQ^-.

We relate the orientation of angles to the halfplanes defined by a line, as in Figure 2.1.

Proposition 2.2.3. *Let X, Y, P, and Q be points in $\mathbb{R}^2$. Then X and Y are in the same halfplane PQ^+ or PQ^- if and only if the angles $\angle PQX$ and $\angle PQY$ have the same orientation.*

Proof. Let $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ in the basis $\{\hat{r}_1, \hat{r}_2\}$ of the previous definition, so both $\hat{r}_1$ and $\hat{r}_2$ are vectors based in P. Then X and Y are in the same halfplane PQ^+ or PQ^- if and only if $\operatorname{sign}(x_2) = \operatorname{sign}(y_2)$.

Note also that $P = (0, 0)$ and $Q = (1, 0)$ in this basis. So the orientation of the angle $\angle PQX$ is

$$\operatorname{sign}\big((q_1 - p_1)(q_2 - x_2) - (q_2 - p_2)(q_1 - x_1)\big) = \operatorname{sign}(-x_2),$$

and, similarly, the orientation of $\angle PQY$ is $\operatorname{sign}(-y_2)$. Hence X and Y are in the same halfplane PQ^+ or PQ^- if and only if the angles $\angle PQX$ and PQY have the same orientation. ∎

Recall the classical definition of a convex polygon.

Definition 2.2.4. A polygon $\mathcal{A}$ is called *convex* if for any two points A and B in the interior of $\mathcal{A}$, the line segment AB is also contained in the interior of $\mathcal{A}$.

The following proposition introduces an equivalent definition of convexity, based on orientation. For clarity, we use the notation $(A_iA_{i+1})^+$ to denote a halfplane.

Proposition 2.2.5. *Let $\mathcal{A}$ be the polygon with vertices $A_1 \dots A_n$ in $\mathbb{R}^2$ and edges*

$$A_1A_2, \dots, A_{n-1}A_n, A_nA_1.$$

Then the polygon $\mathcal{A}$ is convex if either $(A_nA_1)^+$ and each halfplane $(A_iA_{i+1})^+$, for $i = 1, \dots, n-1$, contains every point $A_1, \dots, A_n$, or each halfplane $(A_iA_{i+1})^-$, for $i = 1, \dots, n-1$, and $(A_nA_1)^-$ contains every point $A_1, \dots, A_n$.

Proof. Assume every halfplane $(A_iA_{i+1})^+$, for $i = 1, \dots, n-1$, and $A_nA_1^+$ contains every point $A_1, \dots, A_n$ (if the points are contained instead in the halfplanes $A_iA_{i+1}^-$ then we relabel $A_i \mapsto A_{n-i+1}$). Let B and C be two points in the interior of $\mathcal{A}$. Then both B and C are in every halfplane $(A_iA_{i+1})^+$, and so the line segment BC is also contained in every halfplane $(A_iA_{i+1})^+$.

Hence $\mathcal{A}$ is convex.

■

Proposition 2.2.6. $\mathrm{SL}(2, \mathbb{Z})$ *transformations preserve the convexity of polygons.*

Proof. This follows as a corollary to Propositions 2.2.1 and 2.2.3. ■

Example 16. Let $\mathcal{P}$ be the pentagon with vertices

$$(4, 1), \quad (4, 2), \quad (3, 4), \quad (1, 5), \quad (2, 3),$$

and let M be the $\mathrm{SL}(2, \mathbb{Z})$ matrix given by

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Then $\mathcal{P}$ and $M\mathcal{P}$ are shown in Figure 2.2, and both are convex.

Definition (Convexity of broken lines). Let $\mathcal{A} = \dots A_{-1}A_0A_1 \dots$ be an infinite broken line. We call $\mathcal{A}$ *convex* if either each halfplane $(A_iA_{i+1})^+$ contains every point A_i for $i \in \mathbb{Z}$, or each halfplane $A_iA_{i+1}^-$ contains every point A_i for $i \in \mathbb{Z}$.

Now let $\mathcal{A} = A_0A_1 \dots A_n$ be a finite broken line. We call $\mathcal{A}$ *convex* if either each halfplane $(A_iA_{i+1})^+$ contains every point A_i for $i = 0, \dots, n-1$, or each halfplane $A_iA_{i+1}^-$ contains every point A_i for $i = 0, \dots, n-1$.

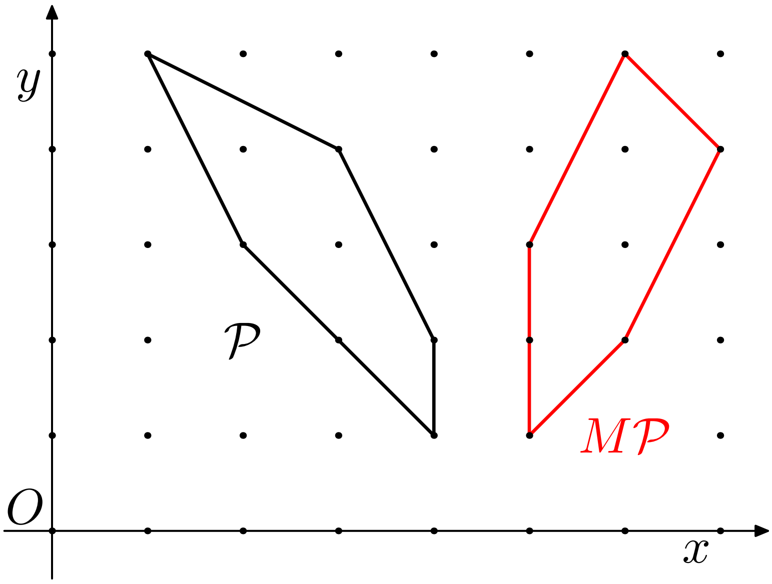

Figure 2.2: The pentagons $\mathcal{P}$ (in black) and $M\mathcal{P}$ (in red).

Remark 2.2.7. Note that $\mathcal{A}$ is not convex if only some halfplanes $(A_iA_{i+1})^+$ contain every point A_j, while other halfplanes $A_iA_{i+1}^-$ contain every point A_j. The orientation of every halfplane must be consistent.

Corollary 2.2.8. An $\mathrm{SL}(2,\mathbb{Z})$ transformation preserves the convexity of infinite broken lines.

The interior of a triangle can be thought of as the intersection of three half planes defined by the sides of the triangle, see Figure 2.3.

Definition 2.2.9. An *empty triangle* is a triangle whose vertices are lattice points, but contains no lattice points in the interior.

Corollary 2.2.10. An $\mathrm{SL}(2,\mathbb{Z})$ transformation sends any empty triangle to an empty triangle.

We relate the $\mathrm{SL}(2,\mathbb{Z})$ equivalence of line segments whose endpoints contain the origin to their integer length. Here and below let $O=(0,0)$ denote the origin.

Proposition 2.2.11. *Let A and B be two integer points. The line segments OA and OB are $\mathrm{SL}(2,\mathbb{Z})$ equivalent if and only if the integer lengths $\mathrm{l}\ell(OA)$ and $\mathrm{l}\ell(OB)$ are equal.*

Proof. For any integer point V we show that OV is $\mathrm{SL}(2,\mathbb{Z})$ equivalent to the line OW, with $W=(n,0)$, if and only if $\mathrm{l}\ell(OV)=n$. Let $V=(x,y)$.

Assume $\gcd(x,y)=n$. Let a and c be integers such that $(x,y)=(an,cn)$ with $\gcd(a,c)=1$. Then there exist integers b and d such that $ad-bc=1$ by Bézout's identity. Define the matrix M by

$$M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

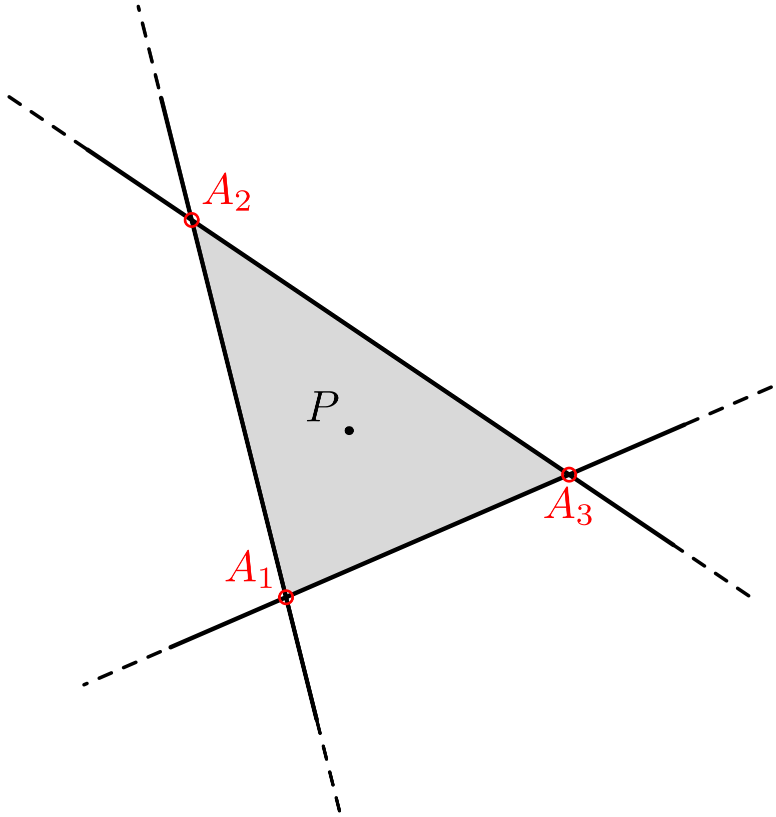

Figure 2.3: A point P in the interior of a triangle $\triangle A_1A_2A_3$.

Then $M \in \mathrm{SL}(2,\mathbb{Z})$ and $Mw = v$.

Assume there exists a matrix M in $\mathrm{SL}(2,\mathbb{Z})$ defined by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

such that $Mw = v$. Then $v = (an, cn)$. Since $\gcd(a,c) = 1$ we have that $\gcd(an, cn) = n$. This completes the proof. ■

Corollary 2.2.12. An $\mathrm{Aff}(2,\mathbb{Z})$ transformation preserves integer length of integer segments.

We relate the affine equivalence of line segments to their integer length.

Proposition 2.2.13. *Two line segments AB and CD are* $\mathrm{Aff}(2,\mathbb{Z})$ *equivalent if and only if* $\mathrm{l}\ell(AB) = \mathrm{l}\ell(CD)$.

Proof. We show that any line segment AB is $\mathrm{Aff}(2,\mathbb{Z})$ equivalent to the segment OW with $W = (n,0)$ if and only if $\mathrm{l}\ell(AB) = n$.

Let $G \in \mathrm{Aff}(2,\mathbb{Z})$ be a transformation sending A to O. Then G sends B to the point GB, and we have a line segment $O(GB)$, which is $\mathrm{SL}(2,\mathbb{Z})$ equivalent to OW if and only if $\mathrm{l}\ell\big(O(GB)\big) = n$, by Proposition 2.2.11. Hence, by Lemma 2.2.12, AB is $\mathrm{Aff}(2,\mathbb{Z})$ equivalent to CD if and only if $\mathrm{l}\ell(AB) = \mathrm{l}\ell(CD)$. ■

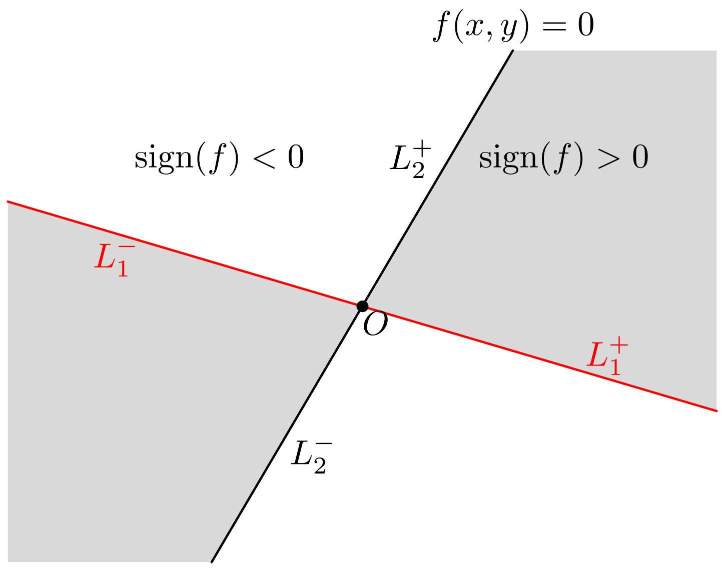

Figure 2.4: The rays L_1^+, L_1^-, L_2^+, and L_2^-.

Remark 2.2.14. Note that two line segments are not necessarily $\mathrm{SL}(2,\mathbb{Z})$ equivalent when they have the same integer length.

2.2.2 $\mathrm{SL}(2,\mathbb{Z})$ transformations of forms

In this subsection we take a *form* $f(x, y)$ to mean an indefinite binary quadratic form, where the roots to the polynomial $f(x, 1)$ are irrational. The solutions to $f(x, y) = 0$ are four rays L_1^+, L_1^-, L_2^+, and L_2^-. These are defined so that both $L_1^+ \cup L_1^-$ and $L_2^+ \cup L_2^-$ form lines, and L_2^+ and L_1^+ form a quadrant in which the form attains positive values, with L_2^+ counter clockwise to L_1^+, as per Figure 2.4.

$\mathrm{SL}(2,\mathbb{Z})$ transformations of sails of forms

Definition 2.2.15. Let V_f^+ denote a sail of the form f enclosed by the rays L_1^+ and L_2^+. Let V_f^- denote the sail of f enclosed by the rays L_2^+ and L_1^-. This is shown in Figure 2.5.

The sail V_f^+ is a convex infinite broken line. The interior region of V_f^+ and the lines given by $f(x, y) = 0$ has no integer points.

We label the quadrant enclosed by L_1^+ and L_2^+ as $L_1^+L_2^+$. We label the other quadrants similarly. Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an $\mathrm{SL}(2,\mathbb{Z})$ transformation. Each quadrant is convex, and so every integer point in $L_1^+L_2^+$ is mapped by M to a quadrant $ML_1^+ML_2^+$, where ML_1^+ and ML_2^+ are rays in the solution to $g(x, y) = f(M^{-1}) = f(dx - by, -cx + ay)$ (we show this in Proposition 2.2.16 below).

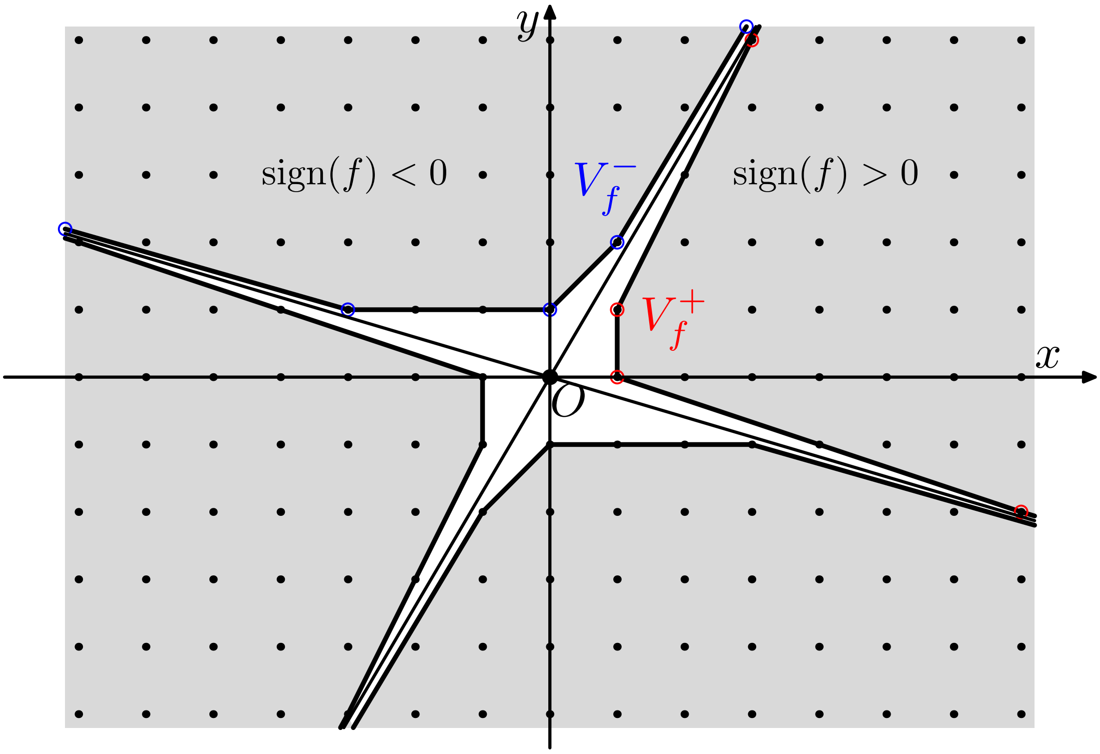

Figure 2.5: V_f^+ and V_f^-.

Similarly, the sail V_f^+ is mapped to an infinite convex broken line, labelled by MV_f^+, where if

$$V_f^+ = \dots A_{n-1}A_nA_{n+1}\dots$$

then

$$MV_f^+ = \dots MA_{n-1}MA_nMA_{n+1}\dots.$$

This line must be within the quadrant $ML_1^+ML_2^+$ by Corollary 2.2.8. The interior region enclosed by MV_f^+ and $ML_1^+ML_2^+$ must be empty by the same reason, and so MV_f^+ is a sail of the form $g(x, y) = f(dx - by, -cx + ay)$.

Hence we have the following proposition.

Proposition 2.2.16. *Let f be a form. Then any* $\mathrm{SL}(2, \mathbb{Z})$ *transformation*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

sends the sails V_f^+ and V_f^- to sails of the form $f(dx - by, -cx + ay)$.

Proof. We only need to verify that M sends the lines in the solutions of the equation $f(x, y) = 0$ to the lines in the solutions of $f(dx - by, -cx + ay) = 0$.

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be points such that O, P_1, and P_2 are not collinear, and $f(P_1) = f(P_2) = 0$. $\mathrm{SL}(2, \mathbb{Z})$ transformations preserve convexity, and so O, MP_1, and MP_2 are not collinear.

Let $g = f(dx - by, -cx + ay)$. Then

$$\begin{aligned} g(MP_1) &= f\big(d(ax_1 + by_1) - b(cx_1 + dy_1), -c(ax_1 + by_1) + a(cx_1 + dy_1)\big) \\ &= f(dax_1 + dby_1 - bcx_1 - bdy_1, -acx_1 - bcy_1 + acx_1 + ady_1) \\ &= f\big(x_1(ad - bc), y_1(ad - bc)\big) \\ &= f(P_1) \\ &= 0. \end{aligned}$$

Similar calculations show that $g(MP_2) = 0$. Hence the solutions of the form $f(x, y) = 0$ get mapped by M to solutions of the form $g(x, y) = 0$. ■

Let $f\big(V_f^+\big)$ be the set of values that f attains on the vertices in V_f^+, so

$$f\big(V_f^+\big) = \big\{f(a, b) \mid (a, b) \in V_f^+\big\}.$$

Define $f\big(V_f^-\big)$ similarly for V_f^-. Let $MV_f^+ = \big\{Mv \mid v \in V_f^+\big\}$, the shift of the vertices of V_f^+ by M. The following proposition relates the values of a form f at its sail to the values of the sails of the form $g = f(M^{-1})$.

Proposition 2.2.17.

$$f\big(V_f^\pm\big) = g\big(V_g^\pm\big).$$

Proof. Let $P = (x_1, y_1)$ be a point on a sail of f. Then $MP = (ax_1+by_1, cx_1+dy_1)$ is a point on the sail of g, as per Proposition 2.2.16. Then

$$\begin{aligned} g(MP) &= f\big(d(ax_1 + by_1) - b(cx_1 + dy_1), -c(ax_1 + by_1) + a(cx_1 + dy_1)\big) \\ &= f(adx_1 - bcx_1, ady_1 - bcy_1) \\ &= f(P). \end{aligned}$$

■

Reduced forms

Let us define reduced forms.

Definition 2.2.18. We call a form f *reduced* if it decomposes into two linear factors

$$f(x, y) = \lambda(y - r_1x)(y - r_2x),$$

where λ is a non zero real number, $1 < r_1$, and $-1 < r_2 < 0$.

We have the following result showing the LLS sequence of reduced forms from the book by O. Karpenkov [52, Theorems 3.1 and 3.4].

Proposition 2.2.19. *Let $f(x, y) = \lambda(y - r_1x)(y - r_2x)$ be a reduced form form with $1 < r_1$ and $-1 < r_2 < 0$. Let the continued fractions of r_1 and r_2 be*

$$\begin{aligned} r_1 &= [a_1; a_2 : \ldots], \\ r_2 &= -[0; b_1 : b_2 : \ldots]. \end{aligned}$$

Then the LLS sequence of f is

$$\ldots, b_2, b_1, a_1, a_2, \ldots$$

We define an $\mathrm{SL}(2,\mathbb{Z})$ matrix specific to a form f.

Definition 2.2.20. Let $\alpha = (a_1, \dots, a_n)$ be a finite even sequence of positive integers. Define the form f_α and matrix M_α by

$$f_\alpha(x,y) = K_1^{n-1}(\alpha)x^2 + \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)xy - K_2^n(\alpha)y^2,$$
$$M_\alpha = \begin{pmatrix} K_2^{n-1}(\alpha) & K_2^n(\alpha) \\ K_1^{n-1}(\alpha) & K_1^n(\alpha) \end{pmatrix}.$$

Remark 2.2.21. From Proposition 2.1.8 (*iii*) we have that $\det\big(M_\alpha\big) = 1$ since n is even. Note that we may write f_α in terms of numerators and denominators of continued fractions. Let

$$[a_1; a_2 : \ldots : a_n] = \frac{P_1}{Q_1},$$
$$[a_1; a_2 : \ldots : a_{n-1}] = \frac{P_2}{Q_2}.$$

Then

$$f_\alpha(x,y) = P_2x^2 + (P_1 - Q_2)xy - Q_1y^2.$$

Proposition 2.2.22. *The form f_α is reduced. The LLS sequence of f_α is periodic, with period α.*

Proof. Recall for a sequence of real numbers $(a_1, \dots, a_n, x)$ that

$$K_1^{n+1}(a_1, \dots, a_n, x) = K_1^n(a_1, \dots, a_n)x + K_1^{n-1}(a_1, \dots, a_{n-1}).$$

For an infinite regular continued fraction with period α, with $\alpha = (a_1, \dots, a_n)$ we have

$$\begin{aligned}[\langle\alpha\rangle] = [\langle a_1, \dots, a_n\rangle] &= [a_1; \dots : a_n : \langle a_1, \dots, a_n\rangle] \\ &= \big[a_1; \dots : a_n : [\langle\alpha\rangle]\big] \\ &= \frac{K_1^n(\alpha)[\langle\alpha\rangle] + K_1^{n-1}(\alpha)}{K_2^n(\alpha)[\langle\alpha\rangle] + K_2^{n-1}(\alpha)}\end{aligned}$$

and

$$K_2^n(\alpha)[\langle\alpha\rangle]^2 + K_2^{n-1}(\alpha)[\langle\alpha\rangle] = K_1^n(\alpha)[\langle\alpha\rangle] + K_1^{n-1}(\alpha).$$

Hence $[\langle\alpha\rangle]$ is a solution to the quadratic equation

$$K_1^{n-1}(\alpha) + \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)y - K_2^n(\alpha)y^2 = 0,$$

and $\big(1, [\langle\alpha\rangle]\big)$ is a solution to $f_\alpha(x,y) = 0$, where

$$f_\alpha(x,y) = K_1^{n-1}(\alpha)x^2 + \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)xy - K_2^n(\alpha)y^2.$$

Similarly we have that

$$\begin{aligned}[\langle\overline{\alpha}\rangle] = [\langle a_n, \dots, a_1\rangle] &= [a_n; \dots : a_1 : \langle a_n, \dots, a_1\rangle] \\ &= \big[a_n; \dots : a_1 : [\langle\overline{\alpha}\rangle]\big] \\ &= \frac{K_1^n(\overline{\alpha})[\langle\overline{\alpha}\rangle] + K_1^{n-1}(\overline{\alpha})}{K_2^n(\overline{\alpha})[\langle\overline{\alpha}\rangle] + K_2^{n-1}(\overline{\alpha})}.\end{aligned}$$

Recall from Proposition 2.1.8 (*ii*) that $K_1^n(\alpha) = K_1^n(\overline{\alpha})$, so

$$K_1^{n-1}(\alpha) = K_2^n(\overline{\alpha}) \qquad \text{and} \qquad K_2^n(\alpha) = K_1^{n-1}(\overline{\alpha}).$$

From this symmetric property of continuants we have that

$$[\langle\overline{\alpha}\rangle] = \frac{K_1^n(\overline{\alpha})[\langle\overline{\alpha}\rangle] + K_1^{n-1}(\overline{\alpha})}{K_2^n(\overline{\alpha})[\langle\overline{\alpha}\rangle] + K_2^{n-1}(\overline{\alpha})} = \frac{K_1^n(\alpha)[\langle\overline{\alpha}\rangle] + K_2^n(\alpha)}{K_1^{n-1}(\alpha)[\langle\overline{\alpha}\rangle] + K_2^{n-1}(\alpha)}$$

Hence $[\langle\overline{\alpha}\rangle]$ is a solution to the quadratic equation

$$K_1^{n-1}(\alpha)x^2 - \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)x - K_2^n(\alpha) = 0,$$

and $\big([\langle\overline{\alpha}\rangle], -1\big)$ is a solution to $f_\alpha(x,y) = 0$, where

$$f_\alpha(x,y) = K_1^{n-1}(\alpha)x^2 + \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)xy - K_2^n(\alpha)y^2.$$

From this we have that

$$f_\alpha(x,y) = -K_2^n(\alpha)\big(y - [\langle\alpha\rangle]x\big)\big(y + [0;\langle\overline{\alpha}\rangle]x\big).$$

Hence f_α is reduced, and by Proposition 2.2.19 we have that the LLS sequence of f_α is periodic with period α. ■

The following proposition says how M_α acts on the sails of f_α.

Proposition 2.2.23. *The matrix M_α acts as a permutation on the sails of f_α.*

Proof. This is a corollary to Proposition 2.2.16, with the fact that if

$$f(x,y) = cx^2 + (d-a)xy - by^2$$

then also

$$f(ax+by, cx+dy) = cx^2 + (d-a)xy - by^2.$$

■

Definition 2.2.24. Let V denote an infinite sail with a periodic LLS sequence α, whose period is $(a_1, \ldots, a_n)$. Any finite broken line made up of $n/2$ segments of V is called a *period of the sail V*.

Let α be a periodic LLS sequence.

Corollary 2.2.25. The values that f_α attains on a period of its sail is preserved by M_α.

The value of a form on the vertices of its sail

Now we prove the following proposition about the values that a form attains on the vertices of its sails. Recall the notation for the continuant of a sequence $\alpha = (a_1, \dots, a_n)$

$$\breve{K}(\alpha) = K_1^{n-1}(\alpha).$$

Let $\alpha_i = (\alpha_i, \dots, \alpha_n, \alpha_1, \dots, \alpha_{i-1})$ for $i = 1, \dots, n$. We call α_i the *i-th cyclic shift of α.*

Proposition 2.2.26. *For a form f_α we have that the set of values that f_α attains on a single period of its sail starting from $(1,0)$ are*

$$\{\breve{K}(\alpha_i) \mid \alpha_i \text{ a cyclic shift of } \alpha,\ i \text{ odd}\}.$$

We define matrices that take the sail of f_α to the sail of f_{α_i}.

Definition 2.2.27. For i odd let

$$T_i = \begin{pmatrix} K_{i+1}^{n-1}(\alpha) & K_{i+1}^{n}(\alpha) \\ K_{i}^{n-1}(\alpha) & K_{i}^{n}(\alpha) \end{pmatrix}.$$

For i even let

$$T_i = \begin{pmatrix} -K_1^{i-1}(\alpha) & K_2^{i-1}(\alpha) \\ K_1^{i-2}(\alpha) & -K_2^{i-2}(\alpha) \end{pmatrix}.$$

Example 17. Recall that $K_2^0(\alpha) = 0$ for any α. Then for $\alpha = (1, 2, 3, 4)$ we have

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 4 \\ 3 & 13 \end{pmatrix}, \quad T_4 = \begin{pmatrix} -10 & 7 \\ 3 & -2 \end{pmatrix}.$$

Remark 2.2.28. Note that for i odd we have

$$\det(T_i) = K_i^n(\alpha)K_{i+1}^{n-1}(\alpha) - K_i^{n-1}(\alpha)K_{i+1}^n(\alpha) = 1,$$

and for i even we have

$$\det(T_i) = K_1^{i-1}(\alpha)K_2^{i-2}(\alpha) - K_1^{i-2}(\alpha)K_2^{i-1}(\alpha) = -1,$$

by Proposition 2.1.8 (*iii*).

Proposition 2.2.29. *The matrix T_i sends the sails of f_α to the sails of f_{α_i}. That is,*

$$T_i M_\alpha T_i^{-1} = M_{\alpha_i}.$$

Proof. For i odd we have that $T_i M_\alpha$ is equal to

$$\begin{pmatrix} K_{i+1}^{n-1}(\alpha)K_2^{n-1}(\alpha)+K_{i+1}^{n}(\alpha)K_1^{n-1}(\alpha) & K_{i+1}^{n-1}(\alpha)K_2^{n}(\alpha)+K_{i+1}^{n}(\alpha)K_1^{n}(\alpha) \\ K_{i}^{n-1}(\alpha)K_2^{n-1}(\alpha)+K_{i}^{n}(\alpha)K_i^{n-1}(\alpha) & K_{i}^{n-1}(\alpha)K_2^{n}(\alpha)+K_{i}^{n}(\alpha)K_1^{n}(\alpha) \end{pmatrix}.$$

Note that

$$\alpha_i = (a_i, \dots, a_n, a_1, \dots, a_{i-1}),$$

so using the formula of Proposition 2.1.6 we have that $K_1^n(\alpha_i)$ splits according to

$$(\underbrace{\overbrace{a_i,\dots,a_{n-1}}^{K_i^{n-1}(\alpha)},a_n}_{K_i^n(\alpha)},\underbrace{a_1,\overbrace{a_2,\dots,a_{i-1}}^{K_2^{i-1}(\alpha)}}_{K_1^{i-1}(\alpha)}),$$

and so

$$K_1^n(\alpha_i) = K_i^n(\alpha)K_1^{i-1}(\alpha) + K_i^{n-1}(\alpha)K_2^{i-1}(\alpha).$$

Similarly we have

$$\begin{aligned}K_2^n(\alpha_i) &= K_{i+1}^n(\alpha)K_1^{i-1}(\alpha) + K_{i+1}^{n-1}(\alpha)K_2^{i-1}(\alpha),\\ K_1^{n-1}(\alpha_i) &= K_i^n(\alpha)K_1^{i-2}(\alpha) + K_i^{n-1}(\alpha)K_2^{i-2}(\alpha),\\ K_2^{n-1}(\alpha_i) &= K_{i+1}^n(\alpha)K_1^{i-2}(\alpha) + K_{i+1}^{n-1}(\alpha)K_2^{i-2}(\alpha).\end{aligned}$$

Using this we have that M_{α_i} is equal to

$$\begin{pmatrix} K_{i+1}^n(\alpha)K_1^{i-2}(\alpha) + K_{i+1}^{n-1}(\alpha)K_2^{i-2}(\alpha) & K_{i+1}^n(\alpha)K_1^{i-1}(\alpha) + K_{i+1}^{n-1}(\alpha)K_2^{i-1}(\alpha) \\ K_i^n(\alpha)K_1^{i-2}(\alpha) + K_i^{n-1}(\alpha)K_2^{i-2}(\alpha) & K_1^n(\alpha)K_1^{i-1}(\alpha) + K_i^{n-1}(\alpha)K_2^{i-1}(\alpha)\end{pmatrix}.$$

The upper left element of $M_{\alpha_i}T_i$ is

$$\begin{aligned}&\Big(K_{i+1}^n(\alpha)K_1^{i-2}(\alpha) + K_{i+1}^{n-1}(\alpha)K_2^{i-2}(\alpha)\Big)K_{i+1}^{n-1}(\alpha)\\ &+\Big(K_{i+1}^n(\alpha)K_1^{i-1}(\alpha) + K_{i+1}^{n-1}(\alpha)K_2^{i-1}(\alpha)\Big)K_i^{n-1}(\alpha)\\ =&K_{i+1}^n(\alpha)\Big(K_1^{i-2}(\alpha)K_{i+1}^{n-1}(\alpha) + K_1^{i-1}(\alpha)K_i^{n-1}(\alpha)\Big)\\ &+K_{i+1}^{n-1}(\alpha)\Big(K_2^{i-1}(\alpha)K_i^{n-1}(\alpha) + K_2^{i-2}(\alpha)K_{i+1}^{n-1}(\alpha)\Big).\end{aligned}$$

Recall from Proposition 2.1.6 that

$$\begin{aligned}K_1^{n-1}(\alpha) &= K_1^{i-2}(\alpha)K_{i+1}^{n-1}(\alpha) + K_1^{i-1}(\alpha)K_i^{n-1}(\alpha),\\ K_2^{n-1}(\alpha) &= K_2^{i-1}(\alpha)K_i^{n-1}(\alpha) + K_2^{i-2}(\alpha)K_{i+1}^{n-1}(\alpha),\end{aligned}$$

and so the upper left element of $M_{\alpha_i}T_i$ is

$$K_{i+1}^n(\alpha)K_1^{n-1}(\alpha) + K_{i+1}^{n-1}(\alpha)K_2^{n-1}(\alpha).$$

This is the upper left element of T_iM_α. Similar calculations show the other elements are equal, as well for the i even case. ■

Corollary 2.2.30. The values that f_α attains on its positive sail are the values $f_{\alpha_i}(1,0)$, for odd i. The values that f_α attains on its negative sail are the values $-f_{\alpha_i}(1,0)$, for even i.

Example 18. Let $\alpha = (1,2,3,4)$. Then

$$\begin{aligned}f_\alpha(x,y) = f_{\alpha_1}(x,y) &= 10x^2 + 36xy - 30y^2,\\ f_{\alpha_2}(x,y) &= 30x^2 + 24xy - 16y^2,\\ f_{\alpha_3}(x,y) &= 16x^2 + 40xy - 14y^2,\\ f_{\alpha_4}(x,y) &= 14x^2 + 44xy - 10y^2.\end{aligned}$$

A period of the sail of α containing the point $(1,0)$ (the *positive sail*) contains the points $(1,0)$ and $(1,1)$. A period of the sail of α containing the point $(0,1)$ (the *negative sail*) contains the points $(0,1)$ and $(2,3)$. The values that f_α attains on its positive sail are $\{10,16\}$. The values that f_α attains on its negative sail are $\{-30,-14\}$.

Remark 2.2.31. The values that f_α attains on its sails are then

$$\begin{aligned} f_\alpha\big(V_f^{\pm}\big) &= \{\breve{K}(\alpha_i), -K_2^n(\alpha_i) \mid i \text{ odd}\},\\ &= \{-\breve{K}(\alpha_i), K_2^n(\alpha_i) \mid i \text{ even}\},\\ &= \{-\breve{K}(\alpha_i), \breve{K}(\alpha_j) \mid i \text{ even},\ j \text{ odd}\},\\ &= \{K_2^n(\alpha_i), -K_2^n(\alpha_j) \mid i \text{ even},\ j \text{ odd}\}. \end{aligned}$$

If any cyclic shift α_i of α is palindromic then the values that f_α attains on its positive sail are the absolute values of the values f_α attains on its negative sail.

$$\{\breve{K}(\alpha_i) \mid i \text{ even}\} = \{\breve{K}(\alpha_i) \mid i \text{ odd}\}.$$

Proposition 2.2.26 follows as a corollary to Proposition 2.2.29.

Example 19. Consider the sequence $\alpha = (1,2,3,4)$. Then

$$\begin{aligned} 10 &= \breve{K}(\alpha_1) = K_1^3(1,2,3,4) = K_3(1,2,3) = K_2^n(4,1,2,3) = K_2^n(\alpha_4),\\ 16 &= \breve{K}(\alpha_3) = K_1^3(3,4,1,2) = K_3(3,4,1) = K_2^n(2,3,4,1) = K_2^n(\alpha_2). \end{aligned}$$

Also

$$\begin{aligned} 30 &= \breve{K}(\alpha_2) = K_1^3(2,3,4,1) = K_3(2,3,4) = K_2^n(1,2,3,4) = K_2^n(\alpha_1),\\ 14 &= \breve{K}(\alpha_2) = K_1^3(4,1,2,3) = K_3(4,1,2) = K_2^n(3,4,1,2) = K_2^n(\alpha_3). \end{aligned}$$

Geometric interpretation of T_i

The definition of T_i comes from the following observation. The line segment $(1,0)$ to $(1,a_1)$ on the sail of α has integer length given by a_1. We define the $\mathrm{SL}(2,\mathbb{Z})$ matrix sending the points $(1,0)$ and $(1,a_1)$ to the vertices on the sail of α_i defining the line segment whose integer length is given by a_1. These vertices for i odd are

$$\begin{pmatrix} K_2^{n-i}(\alpha_i) \\ K_1^{n-i}(\alpha_i) \end{pmatrix}, \qquad \begin{pmatrix} K_2^{n-i+2}(\alpha_i) \\ K_1^{n-i+2}(\alpha_i) \end{pmatrix},$$

and for i even we have

$$\begin{pmatrix} -K^n_{n-i+2}(\alpha_i) \\ K^{n-1}_{n-i+2}(\alpha_i) \end{pmatrix}, \qquad \begin{pmatrix} -K^{n-2}_{n-i+2}(\alpha_i) \\ K^{n-3}_{n-i+2}(\alpha_i) \end{pmatrix}.$$

See for example Figure 2.6, which shows the sail for $\alpha = (1,1,2,2)$ being mapped to $\alpha_3 = (2,2,1,1)$, and Figure 2.7 which shows the sail for α being mapped to

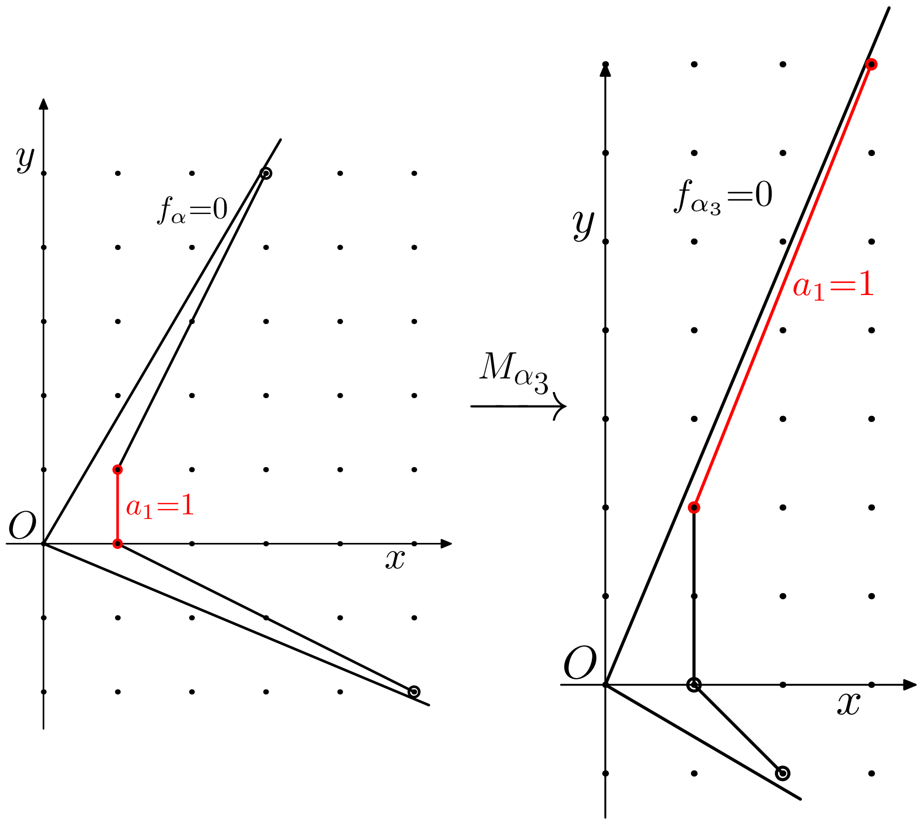

Figure 2.6: The sail for $\alpha = (1,1,2,2)$ being mapped to the sail of $\alpha_3 = (2,2,1,1)$.

$\alpha_2 = (1,2,2,1)$. The transformation matrix in each case is derived by mapping the points $(1,0)$ and $(1,a_1) = (1,2)$ to the corresponding vertices for the line segment given by a_1. These are M_{α_3} and M_{α_1}, where

$$M_{\alpha_3}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}, \quad M_{\alpha_3}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\7\end{pmatrix}$$

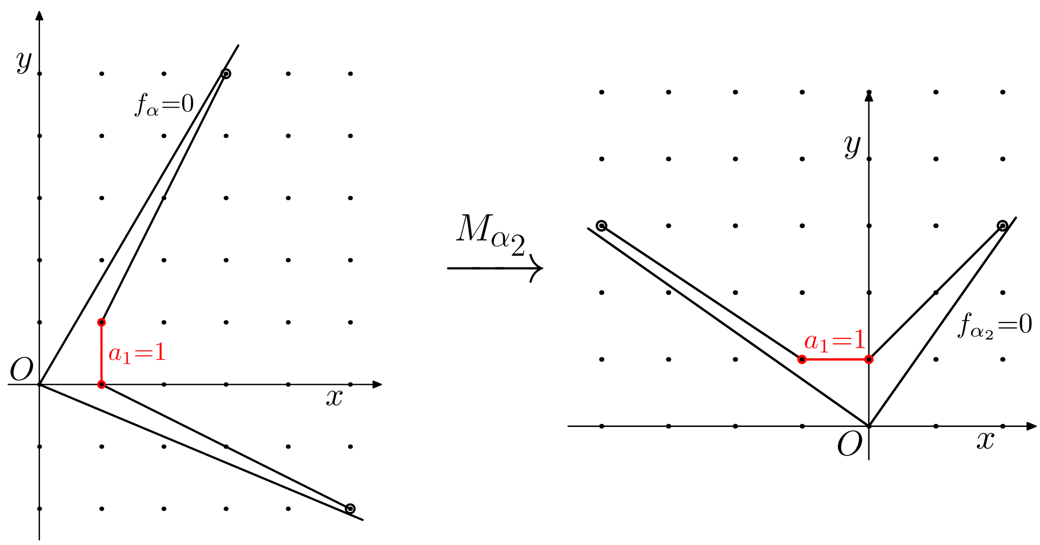

Figure 2.7: The sail for $\alpha = (1,1,2,2)$ being mapped to the sail of $\alpha_2 = (1,2,2,1)$.

and

$$M_{\alpha_2}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}, \quad M_{\alpha_2}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}.$$

Hence

$$M_{\alpha_3}=\begin{pmatrix}1&2\\2&5\end{pmatrix}, \quad M_{\alpha_2}=\begin{pmatrix}-1&1\\1&0\end{pmatrix}.$$

We note that the matrices T_i are in $\mathrm{GL}(2,\mathbb{Z})$.

Lemma 2.2.32. For the matrices T_i we have that

$$\det(T_i)=(-1)^{i+1}.$$

This is clear from the relation for continuants for a sequence of positive integers $\alpha=(a_1,\dots,a_n)$

$$K_1^n(\alpha)K_2^{n-1}(\alpha)-K_2^n(\alpha)K_1^{n-1}(\alpha)=(-1)^n.$$

In particular, T_i is in $\mathrm{SL}(2,\mathbb{Z})$ if i is odd, and $\mathrm{GL}(2,\mathbb{Z})$ otherwise.

Recall the Markov minimum of a form f

$$m(f)=\inf_{\mathbb{Z}^2\setminus\{(0,0)\}}|f(x,y)|.$$

Proposition 2.2.33. *If the Markov minimum of a form f is attained at a point $v\in\mathbb{Z}^2$ then v is a vertex of a sail of f.*

Proof. Let $m(f)=|m|$, for some $|m|\in\mathbb{Z}_+$. Assume $m>0$ (otherwise consider the form $-f$, which has the same sails). Let $0<n<1$. The curve $f(x,y)=n$ is a hyperbola contained between the solutions of $f(x,y)=0$ and the sail of f. It is convex. There is some real number $N>0$ for which the hyperbola $f(x,y)=N$ contains an integer point, while no other integer point satisfies $0<f(x,y)<N$. Any integer point on $f(x,y)=N$ is on the sail of f, since the curve is convex. Hence $N=m$.

■

Combining Propositions 2.2.26 and 2.2.33 we have the proof of the following statement.

Proposition 2.2.34. *Let f be a form with a periodic LLS sequence, with period $\alpha=(a_1,\dots,a_n)$. Then the Markov minimum of f is a continuant of some cyclic shift of α, so*

$$m(f)=\min\left\{\breve{K}(\alpha_i)\mid i=1,\dots,n\right\}.$$

2.3 Integer geometry and the Markov spectrum

In this section we state a relation between values of forms and the continued fractions of certain broken lines. This relation is given in Theorem 2.3.14. We shall see that a special case of this theorem is the *Perron identity*, a central result going back to Perron [76] linking doubly infinite sequences to the Markov spectrum.

The results in this section are proved in the paper [54] by O. Karpenkov and M. Van Son. The definitions in this section are taken directly from this paper.

We have the definition of finite f-broken lines.

Definition 2.3.1. Consider a form f with linear factors $f_1(x, y) = ax + by$ and $f_2(x, y) = cx + dy$. We call a broken line with vertices $A_0 \dots A_n$ a *finite f-broken line* if the following hold:

- the endpoints $A_0, A_n \neq O$ satisfy $f_1(A_0) = 0$ and $f_2(A_n) = 0$;
- all edges of the broken line have positive length;
- the line $A_{k-1}A_k$ does not pass through the origin for any $k = 1, \dots, n$.

Example 20. The left hand side of Figure 2.8 shows an f-broken line for the form $3y^2 - 2xy - 5x^2$. The shaded area in the right hand side figure is a convex hull of the form $5x^2 - 3xy$. The boundary of this is one of the form's sails, which is also an f-broken line.

Recall the definition of oriented area of a parallelogram from Definition 1.3.10. For points $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ recall that the orientation of the angle $\angle ABC$ is

$$\operatorname{sign}\big(\det(BA, BC)\big) = \operatorname{sign}\left(\det\begin{pmatrix} b_1 - a_1 & b_2 - a_2 \\ b_1 - c_1 & b_2 - c_2 \end{pmatrix}\right).$$

Definition 2.3.2. Let $\mathcal{A} = A_0A_1 \dots A_n$ be a broken line with $A_0, A_n \neq O$. The *signature* of $\mathcal{A}$ with respect to the origin is the sign of the determinant $\det(OA_0, OA_n)$. We denote this by $\operatorname{sign}(\mathcal{A})$.

Remark 2.3.3. This definition is important since a sign change occurs when we change the ordering of the broken line, i.e. whether A_0 or A_n are considered the start point of the broken line.

We define the LSS sequence of a finite f-broken line.

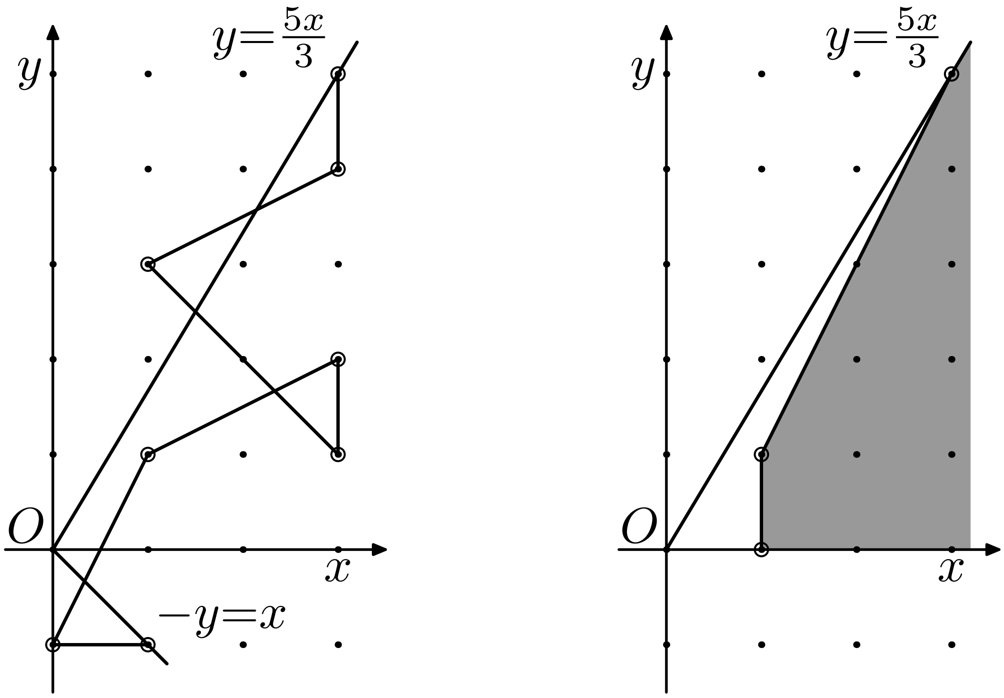

Figure 2.8: **Left hand side:** A broken line for the form $3y^2 - 2xy - 5x^2$.
Right hand side: The sail of the acute angle between $3y = 5x$ and $y = 0$.

Definition 2.3.4. Given an f-broken line $\mathcal{A} = A_0 \dots A_n$ define

$$a_{2k} = \det(OA_k, OA_{k+1}), \quad k = 0, \dots, n;$$
$$a_{2k-1} = \frac{\det(A_kA_{k-1}, A_kA_{k+1})}{a_{2k-2}a_{2k}}, \quad k = 1, \dots, n.$$

We call a_{2k} the *integer scaled length* of the line segment A_kA_{k+1}. The integer sine of the integer angle $\angle A_{k-1}A_kA_{k+1}$ is a_{2k-1}.

The sequence $(a_0, \dots, a_{2n})$ is called the *lattice scaled sine sequence* (or *LSS sequence* for short) for the broken line, and is denoted by LSS($\mathcal{A}$). We call the expression $[a_0; \dots : a_{2n}]$ the *continued fraction for the broken line* $\mathcal{A}$. Note that the values $a_i \neq 0$ may be negative and rational or irrational.

Example 21. We work out the LSS sequence for the sail of the acute angle between $3y = 5x$ and $y = 0$, as in Figure 2.9. Let $A_0 = (1, 0)$, $A_1 = (1, 1)$, and $A_2 = (3, 5)$. Then

$$\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, \quad \text{and} \quad \begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} = 5 - 3 = 2.$$

Further, we have that

$$\frac{\begin{vmatrix} 1-1 & 1-0 \\ 1-3 & 1-5 \end{vmatrix}}{1 \cdot 2} = \frac{\begin{vmatrix} 0 & 1 \\ -2 & -4 \end{vmatrix}}{2} = \frac{2}{2} = 1.$$

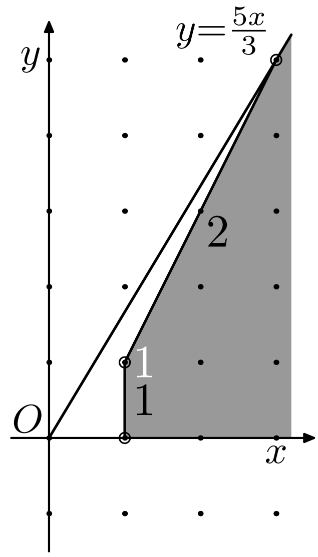

Figure 2.9: The LSS sequence for the sail of the acute angle between $3y = 5x$ and $y = 0$.

Hence the LSS sequence is $(1, 1, 2)$. The broken line between A_0 and A_2 has sign

$$\operatorname{sign}\left(\det\begin{pmatrix}1 & 0\\ 3 & 5\end{pmatrix}\right) = +1.$$

The shaded area in Figure 2.9 is the convex hull of the angle. The boundary of the convex hull is the sail.

Recall Definition 1.3.13 defining the LLS sequence. The following proposition, a proof of which may be found in [52], relates this with the LSS sequence.

Proposition 2.3.5. *For a sail with a finite LLS sequence, the definitions of LLS sequence in Definition 1.3.13 and the LSS sequence in Definition 2.3.4 are equivalent.*

Remark 2.3.6. The LSS sequence of an f-broken line encodes information about the values of the form at the vertices of the broken line, as we see below in Theorem 2.3.14. The LSS sequence is an invariant of the f-broken line under $\mathrm{SL}(2, \mathbb{Z})$ transformations. Hence the continued fractions of the broken lines are $\mathrm{SL}(2, \mathbb{Z})$ invariants, and so their value does not change regardless of the choice of representative from a form's $\mathrm{SL}(2, \mathbb{Z})$ equivalence class.

Remark 2.3.7. Scaling a form f by a non zero real number λ changes the values the form attains at integer points. Such a scaling is an equivalence relation that does not change the solutions to $f(x, y) = 0$ (that is, the arrangement $\mathcal{A}$ of f is also the arrangement for a scaled form λf.). When evaluating the value of f in terms of continued fractions of broken lines we must take care of any scaling factor.

To counter this we multiply our forms by an appropriate scaling factor. Recall that an indefinite binary quadratic form f may be written as the product of two linear factors $f_1(x, y) = ax + by$ and $f_2(x, y) = cx + dy$, where the discriminant of $f(x, y) = f_1(x, y) \cdot f_2(x, y)$ is

$$\Delta(f) = (ad - bc)^2.$$

We scale the values of our form f by $\operatorname{sign}(ac)\sqrt{\Delta(f)}$.

Definition 2.3.8. Let f a the form with linear factors $(ax + by)$ and $(cx + dy)$. The discriminant of f is $\Delta(f) = (ad - bc)^2$. We define the *scaled form* $\tilde{f}(x, y)$ to be

$$\tilde{f}(x, y) = \frac{f(x, y)}{\operatorname{sign}(ac) \cdot \sqrt{\Delta(f)}}.$$

Now we define infinite LSS sequences and sails for irrational angles.

Definition 2.3.9. Let f be a form with linear factors $f_1(x, y) = y - m_1 x$ and $f_2(x, y) = y - m_2 x$ for real numbers m_1 and m_2. An *asymptotic f-broken line* is an infinite in both sides broken line $\ldots A_{-2}A_{-1}A_0A_1A_2\ldots$, for which the following conditions hold (we assume that $A_k = (x_k, y_k)$ for every integer k):

- the two side infinite sequence $\left(\frac{y_n}{x_n}\right)$ converges to different slopes m_1 and m_2 of the linear factors of f as n increases and decreases respectively;
- all edges of the broken line have positive length;
- for $k \in \mathbb{Z}$ the line $A_{k-1}A_k$ does not pass through the origin.

We have the analogue of sign for infinite broken lines.

Definition 2.3.10. Let f be a form with linear factors $f_1(x, y) = y - m_1 x$ and $f_2(x, y) = y - m_2 x$ for real numbers m_1 and m_2. Let $\mathcal{A} = \ldots A_{-2}A_{-1}A_0A_1A_2\ldots$ be an asymptotic f-broken line. Let the slopes $\left(\frac{y_n}{x_n}\right)$ converge to m_2 for $n \to -\infty$, and converge to m_1 for $n \to +\infty$. The *sign of $\mathcal{A}$* with respect to the origin is the following

$$\operatorname{sign}\left(\det\begin{pmatrix}1 & m_2\\ 1 & m_1\end{pmatrix}\right) = \operatorname{sign}(m_1 - m_2).$$

We define the LSS sequence for asymptotic f-broken lines.

Definition 2.3.11. For an asymptotic f-broken line

$$\mathcal{A} = \ldots A_{-2}A_{-1}A_0A_1A_2\ldots$$

let

$$a_{2k} = \det(OA_k, OA_{k+1}), \quad k \in \mathbb{Z};$$
$$a_{2k-1} = \frac{\det(A_kA_{k-1}, A_kA_{k+1})}{a_{2k-2}a_{2k}}, \quad k \in \mathbb{Z}.$$

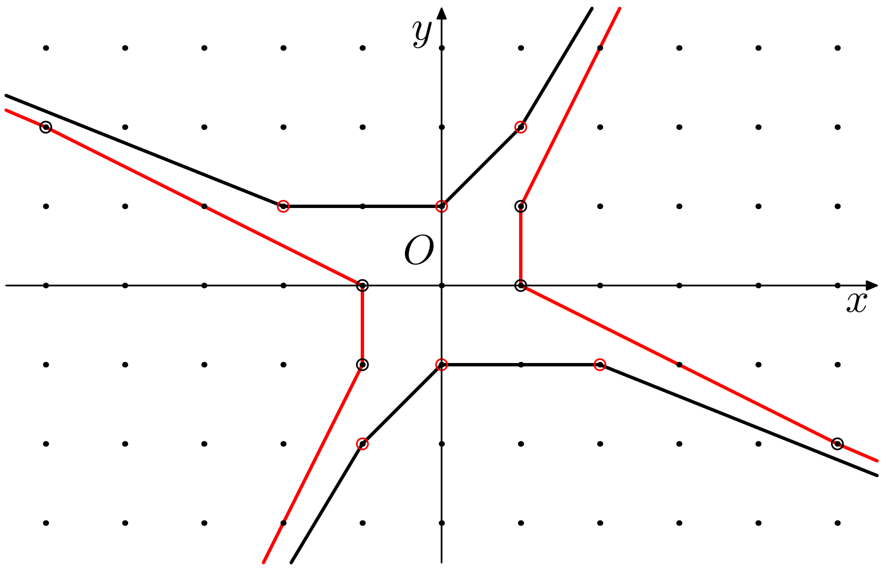

Figure 2.10: The four sails for the form $5x^2+9xy-7y^2$, given by the LSS sequence with period $(1, 1, 2, 2)$.

We call a_{2k} the *integer scaled length* of the line segment A_kA_{k+1}. The integer sine of the integer angle $\angle A_{k-1}A_kA_{k+1}$ is a_{2k-1}.

The sequence $(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2 \ldots)$ is called the *LSS sequence* for the broken line and denoted by $\mathrm{LSS}(\mathcal{A})$.

We relate the *LLS* and *LSS* sequences of sails with the following proposition, a proof of which may be found in [52].

Proposition 2.3.12. *For a sail with an infinite LLS sequence, the LLS sequence (of Definition 1.3.13) and the LSS sequence (of Definition 2.3.11) are equal.*

Remark 2.3.13. The infinite broken lines we work with in the study of the Markov spectrum will be sails of forms. Figure 2.10 shows the sails of the form $5x^2 + 9xy - 7y^2$.

The following theorem from [54] relates continued fractions of broken lines with the values of forms.

Theorem 2.3.14 (O. Karpenkov, M. Van-Son [54])**.**

(*i*) **Finite Case:** *For a form f and scaled form $\tilde{f}$ let $\mathcal{A} = A_0 \ldots A_{n+m}$ be an f-broken line (here n and m are arbitrary positive integers), and let*

$$\mathrm{LSS}(\mathcal{A}) = (a_0, a_1, \ldots, a_{2n+2m}).$$

Then

$$\tilde{f}(A_n) = \frac{\mathrm{sign}(\mathcal{A})}{a_{2n-1} + [0; a_{2n-2} : \ldots : a_0] + [0; a_{2n} : \ldots : a_{2n+2m}]}.$$

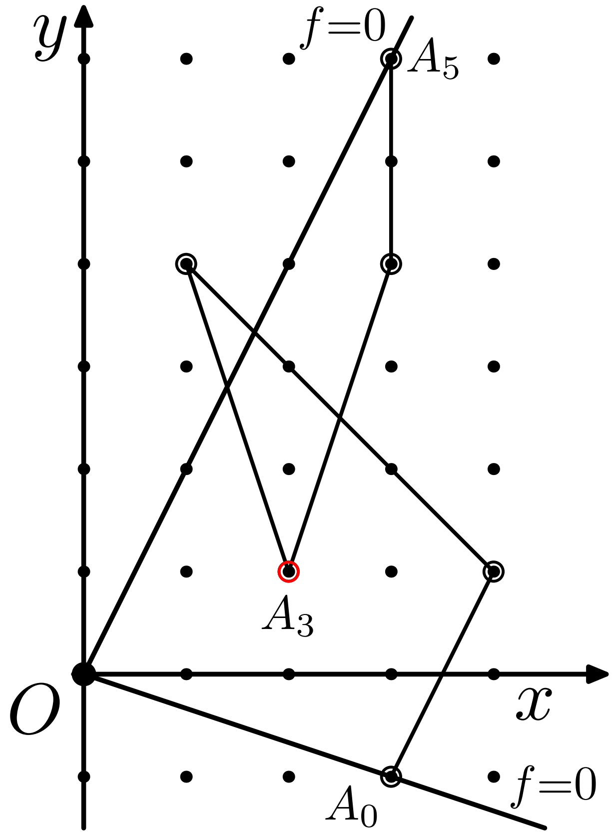

Figure 2.11: The form $(y-2x)(3y+x)$ and broken line $\mathcal{A} = A_0 \ldots A_5$.

(ii) **Infinite Case:** *For a form $f(x, y) = (ax+by)(cx+dy)$ let*

$$\mathcal{A} = \ldots A_{-2}A_{-1}A_0A_1A_2 \ldots$$

be an asymptotic f-broken line, and let

$$\mathrm{LSS}(\mathcal{A}) = (\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots).$$

Assume also that both continued fractions

$$[0; a_{-1} : a_{-2} : \ldots] \quad \textit{and} \quad [0; a_1 : a_2 : \ldots]$$

converge. Then

$$\tilde{f}(A_0) = \frac{\mathrm{sign}(\mathcal{A})}{a_0 + [0; a_{-1} : a_{-2} : \ldots] + [0; a_1 : a_2 : \ldots]}.$$

We prove this theorem in Section 2.4 below.

Example 22. Let $f(x, y) = (y-2x)(3y+x)$, with scaling $-\sqrt{\Delta(f)} = -7$. Let $\mathcal{A} = A_0 \ldots A_5$ be the broken line in Figure 2.11 with vertices

$$\begin{array}{lll} A_0 = (3, -1), & A_1 = (4, 1), & A_2 = (1, 4), \\ A_3 = (2, 1), & A_4 = (3, 4), & A_5 = (3, 6). \end{array}$$

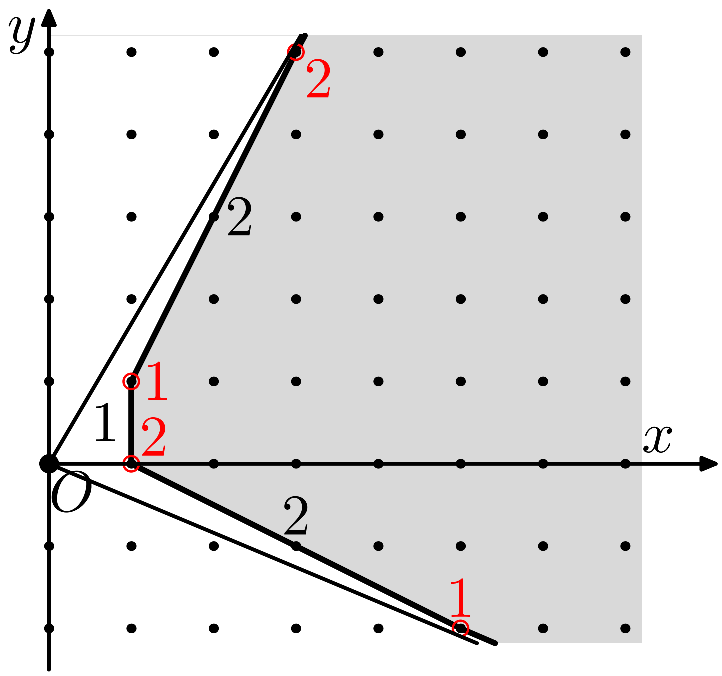

Figure 2.12: Sail and LLS sequence for the form $5x^2 + 9xy - 7y^2$. Integer sines are written in red, and integer lengths are written in black.

Then $\mathcal{A}$ is an f-broken line, with sign given by

$$\operatorname{sign}(\mathcal{A}) = \operatorname{sign}\left(\det\begin{pmatrix} 3 & -1 \\ 3 & 6 \end{pmatrix}\right) = +1.$$

The LSS sequence of $\mathcal{A}$ is

$$\operatorname{LSS}(\mathcal{A}) = \left(7, -\frac{3}{35}, 15, \frac{2}{35}, -7, \frac{6}{35}, 5, -\frac{1}{15}, 6\right).$$

Then the value of the scaled form $\tilde{f}$ at A_2 is given by the theorem as

$$\begin{aligned}\tilde{f}(A_2) &= \frac{\operatorname{sign}(\mathcal{A})}{\frac{2}{25} + \left[0; 15 : -\frac{3}{35} : 7\right] + \left[0; -7 : \frac{6}{35} : 5 : -\frac{1}{15} : 6\right]} \\ &= \frac{+1}{\frac{2}{25} + \frac{2}{65} - \frac{5}{14}} \\ &= -\frac{26}{7}.\end{aligned}$$

Indeed, we have

$$\tilde{f}(A_2) = \frac{f(A_2)}{-7} = -\frac{26}{7}$$

Example 23. Let $f(x, y) = 5x^2 + 9xy - 7y^2$. Let $\mathcal{A} = \ldots A_{-1}A_0A_1 \ldots$ be the sail of f containing the point $(1, 0)$, as in Figure 2.12. Then $\mathcal{A}$ is an f-broken line. The LSS sequence $(a_i)_{i\in\mathbb{Z}}$ is periodic, with period $(1, 1, 2, 2)$. Let a_0 denote the integer sine of angle at $(1, 0)$, and a_2 the integer sine of angle at $(1, 1)$. Then

$$\begin{aligned}(a_i)_{i=1}^{\infty} &= (1, 1, 2, 2, 1, 1, 2, 2, \ldots), \\ (a_i)_{i=0}^{-\infty} &= (2, 2, 1, 1, 2, 2, 1, 1, \ldots).\end{aligned}$$

Since
$$f(x,y) = -7\cdot\left(y-\frac{-9+\sqrt{221}}{-14}x\right)\left(y-\frac{-9-\sqrt{221}}{-14}x\right)$$
we have $\operatorname{sign}(\mathcal{A}) = \operatorname{sign}\left(\frac{\sqrt{221}}{7}\right) = 1$. According to Theorem 2.3.14 the value of f at $(1,0)$ is
$$\begin{aligned} f(1,0) &= \frac{\sqrt{221}}{2+[0;\langle 1,1,2,2\rangle]+[0;\langle 2,1,1,2\rangle]} \\ &= \frac{\sqrt{221}}{2+\dfrac{-9+\sqrt{221}}{10}+\dfrac{-11+\sqrt{221}}{10}} \\ &= 5, \end{aligned}$$
which is correct.

Theorem 2.3.14 is related to the study of the Markov minimum of a form f by Proposition 2.2.33. Note that we have a new proof of Theorem 1.3.19 from Section 1.3. It follows as a corollary to Theorem 2.3.14 and Proposition 2.2.33.

Corollary 2.3.15 (O. Perron)**.** Let $\mathcal{A} = \ldots A_{-2}A_{-1}A_0A_1A_2\ldots$ be a sail of a form f with LLS sequence
$$\mathrm{LLS}(\mathcal{A}) = (\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots).$$
Then $\mathcal{A}$ is an asymptotic f-broken line, and we have the following equality
$$\inf_{\mathbb{Z}^2\setminus O}|f| = \inf_{i\in\mathbb{Z}}\left(\frac{\sqrt{\Delta(f)}}{a_i+[0;a_{i+1}:a_{i+2}:\ldots]+[0;a_{i-1}:a_{i-2}:\ldots]}\right).$$

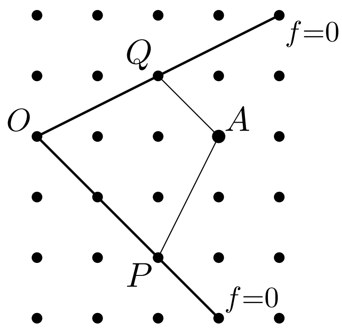

Figure 2.13: The kernel of f and the f-broken line PAQ.

2.4 Proof of Theorem 2.3.14

The work in this section appears in the paper [54], written jointly with O. Karpenkov.

We prove Theorem 2.3.14. We start with three lemmas.

Lemma 2.4.1. Let f be a form. Let $P \neq O$ and $Q \neq O$ annulate distinct linear factors of f. Then for every point A it holds

$$f(A) = \operatorname{sign}(POQ) \cdot \frac{\det(OP, OA) \cdot \det(OA, OQ)}{\det(OP, OQ)} \cdot \sqrt{\Delta(f)}.$$

Example 24. Consider the following binary quadratic form

$$f(x, y) = (x + y)(x - 2y).$$

Let PAQ be an f-broken line, with $P = (2, 1)$, $A = (3, 0)$, and $Q = (2, -2)$, see Figure 2.13. Direct calculations show that

$$\begin{array}{lll} \det(OP, OA) = 6, & \det(OA, OQ) = 3, & \det(OP, OQ) = 6, \\ \operatorname{sign}(POQ) = 1, & f(A) = 9, & \Delta(f) = 9. \end{array}$$

Therefore, we have

$$\begin{aligned} \operatorname{sign}(POQ) \cdot \frac{\det(OP, OA) \cdot \det(OA, OQ)}{\det(OP, OQ)} \cdot \sqrt{\Delta(f)} &= 1 \cdot \frac{6 \cdot 3}{6} \cdot \sqrt{9} = 9 \\ &= f(A). \end{aligned}$$

Proof of Lemma 2.4.1. The statement is straightforward for the form

$$f(x, y) = axy,$$

for a non zero real number a. Assume that $P = (p, 0)$, $Q = (0, q)$, and $A = (x, y)$. Then we have

$$f(A) = axy = \frac{py \cdot qx}{pq} \cdot a = \frac{\det(OP, OA) \cdot \det(OA, OQ)}{\det(OP, OQ)} \cdot \sqrt{\Delta(f)}.$$

For $P=(0,p)$ and $Q=(q,0)$ we have

$$\begin{aligned} f(A) = axy &= \frac{(-px)\cdot(-qx)}{-pq}\cdot a \\ &= -\frac{\det(OP,OA)\cdot\det(OA,OQ)}{\det(OP,OQ)}\cdot\sqrt{\Delta(f)}. \end{aligned}$$

This concludes the proof for the case of f.

The general case follows from the invariance of the expressions of the equality of the lemma under the group of linear area preserving transformations (i.e., whose determinants equal 1) of the plane. ■

Now we prove a particular case of Theorem 2.3.14.

Lemma 2.4.2. Let f be a form. Consider an oriented f-broken line $\mathcal{B}=B_0B_1B_2$ with $\mathrm{LLS}(\mathcal{B})=(b_0,b_1,b_2)$. Then

$$f(B_1)=\frac{\mathrm{sign}(\mathcal{B})\cdot\sqrt{\Delta(f)}}{b_1+[0;b_0]+[0;b_2]}.$$

Proof. Set $B_i=(x_i,y_i)$ for $i=0,1,2$. Then Definition 2.3.4 implies

$$\begin{aligned} b_0 &= \det(OB_0,OB_1)=x_0y_1-x_1y_0, \\ b_2 &= \det(OB_1,OB_2)=x_1y_2-y_1x_2, \\ b_1 &= \frac{\det(B_1B_0,B_1B_2)}{b_0b_2}=\frac{x_0y_2-x_2y_0-x_0y_1+x_1y_0-x_1y_2+y_1x_2}{b_0b_2}. \end{aligned}$$

After a substitution and simplification we get

$$\begin{aligned} \frac{1}{b_1+[0;b_0]+[0;b_2]} &= \frac{(x_0y_1-x_1y_0)(x_1y_2-y_1x_2)}{x_0y_2-x_2y_0} \\ &= \frac{\det(OB_0,OB_1)\cdot\det(OB_1,OB_2)}{\det(OB_1,OB_2)}. \end{aligned}$$

Finally recall that

$$\mathrm{sign}(\mathcal{B})=\mathrm{sign}(B_0B_1B_2).$$

Now Lemma 2.4.2 follows directly from Lemma 2.4.1. ■

For the proof of general case we need the following important result.

Lemma 2.4.3. ([52, Corollary 11.11**, p. 144].)** Consider a broken line $A_0\ldots A_n$ that has the LLS sequence $(a_0,\ldots,a_{2n})$, with $A_0=(1,0)$, $A_1=(1,a_0)$, and $A_n=(x,y)$. Let

$$\alpha=[a_0;a_1:\ldots:a_{2n}]$$

be the corresponding continued fraction for this broken line. Then

$$\frac{y}{x}=\alpha.$$

For the case of an infinite value for $\alpha=[a_0;a_1:\ldots:a_{2n}]$,

$$\frac{x}{y}=0.$$

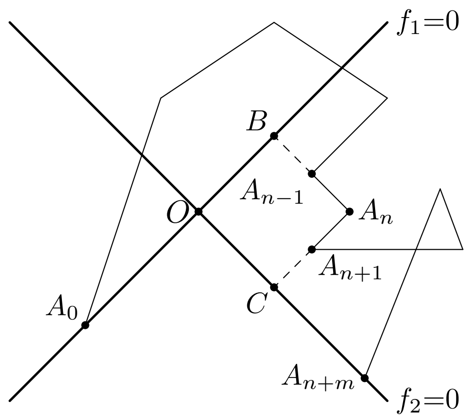

Figure 2.14: The original f-broken line $\mathcal{A}$ and the resulting f-broken line BA_nC.

For a proof of Lemma 2.4.3 we refer to [52]. As a consequence of Lemma 2.4.3 we have the following statement.

Corollary 2.4.4. Consider two broken lines $A_0 \ldots A_n$ and $B_0 \ldots B_m$ that have the LLS sequences $(a_0, \ldots, a_{2n})$, and $(b_0, \ldots, b_{2m})$ respectively. Suppose that the following hold:

- $A_0 = B_0$;
- the points A_n, B_m, and O are in a line;
- the points $A_0 = B_0$, A_1, and B_1 are in a line.

Then

$$[a_0; a_1 : \ldots : a_{2n}] = [b_0; b_1 : \ldots : b_{2n}].$$

Proof. In coordinates of the basis

$$e_1 = OA_0, \qquad e_2 = \frac{A_0A_1}{|A_0A_1||OA_0|}$$

the coincidence of continued fractions follows from Lemma 2.4.3. ■

Now we prove Theorem 2.3.14.

Proof of Theorem 2.3.14. (i) **Finite Case:** Let f be a form. Denote the linear factors of f by f_1 and f_2. Consider an f-broken line $\mathcal{A} = A_0 \ldots A_{n+m}$. Without loss of generality we assume that A_0 and A_{n+m} annulate f_1 and f_2 respectively.

Denote by B the intersection of the line A_nA_{n-1} with the line $f_1 = 0$. Denote by C the intersection of the line A_nA_{n+1} with the line $f_2 = 0$. (See Figure 2.14.) Then the continued fraction for the broken line BA_nC is $[b_0 : a_{2n-1} : b_2]$ for some real numbers b_0 and b_2.

By Corollary 2.4.4 we have

$$b_0 = [a_{2n-2}; \ldots : a_0],$$
$$b_2 = [a_{2n}; \ldots : a_{2n+2m}].$$

By construction

$$\operatorname{sign}(BA_nC) = \operatorname{sign}(\mathcal{A}).$$

Therefore by Lemma 2.4.2 we have

$$\begin{aligned} f(A_n) &= \frac{\operatorname{sign}(BA_nC)\cdot\sqrt{\Delta(f)}}{a_{2n-1}+[0;b_0]+[0;b_2]} \\ &= \frac{\operatorname{sign}(\mathcal{A})\cdot\sqrt{\Delta(f)}}{a_{2n-1}+[0;a_{2n-2}:\ldots:a_0]+[0;a_{2n}:\ldots:a_{2n+2m}]}. \end{aligned}$$

This concludes the proof of the finite case in Theorem 2.3.14.

(ii) **Infinite Case:** Without loss of generality we consider the form

$$f = \lambda f_{\alpha,\beta} = \lambda(y-\alpha x)(y+\beta x),$$

for some nonzero λ and arbitrary $\alpha \neq \beta$. Let $\mathcal{A} = \ldots A_{-2}A_{-1}A_0A_1A_2\ldots$ be an asymptotic f-broken line, where the points $A_k = (x_k, y_k)$ for all integer k. Also we assume that $x_k \neq 0$ for all k (otherwise, switch to another coordinate system, where the last condition holds).

Set

$$\begin{gathered} \mathcal{A}_n = A_{-n}\ldots A_{-2}A_{-1}A_0A_1A_2\ldots A_n; \\ \alpha_n = \frac{y_{-n}}{x_{-n}}; \qquad \beta = \frac{y_n}{x_n}. \end{gathered}$$

First of all, by definition $\operatorname{LLS}(\mathcal{A}_n)$ coincides with $\operatorname{LLS}(\mathcal{A})$ for all admissible entries. Secondly, we immediately have that

$$\lim_{n\to\infty} \lambda f_{\alpha_n,\beta_n}(A_0) = \lambda f_{\alpha,\beta}(A_0).$$

Thirdly, the sequence of signatures stabilizes as n tends to infinity. In other words

$$\lim_{n\to\infty} \operatorname{sign}(\mathcal{A}_n) = \operatorname{sign}(\mathcal{A}).$$

Fourthly,

$$\lim_{n\to\infty} \Delta(\lambda f_{\alpha_n,\beta_n}) = \Delta(\lambda f_{\alpha,\beta})$$

(recall that $\Delta(f)$ is the discriminant of the form f). Finally since both continued fractions

$$[0;a_{-1}:a_{-2}:\ldots], \quad \text{and} \quad [0;a_1:a_2:\ldots]$$

converge and by the above four observations we have

$$\begin{aligned} f(A_0) &= \lim_{n\to\infty} \lambda f_{\alpha_n,\beta_n}(A_0) \\ &= \lim_{n\to\infty} \frac{\operatorname{sign}(\mathcal{A}_n)\cdot\sqrt{\Delta(\lambda f_{\alpha_n,\beta_n})}}{a_0+[0;a_{-1}:a_{-2}:\ldots:a_{2-2n}]+[0;a_1:a_2:\ldots:a_{2n-2}]}. \\ &= \frac{\operatorname{sign}(\mathcal{A})\cdot\sqrt{\Delta(f)}}{a_0+[0;a_{-1}:a_{-2}:\ldots]+[0;a_1:a_2:\ldots]}. \end{aligned}$$

The second equality holds as it holds for the elements in the limits for every positive integer n by Theorem 2.3.14 (i).

This concludes the proof of the infinite case of Theorem 2.3.14. ■

2.5 Markov spectrum and Markov forms

In this section we discuss the discrete subset of the Markov spectrum. We see how it relates to the study of equivalence classes of forms. We also define Cohn matrices and discuss their triple graph structure.

We use several results on forms in our work. Of particular importance is the relation between forms and periodic continued fractions. This theorem was proved in two parts, contributed by L. Euler in [36] and J.L. Lagrange in [60].

We use the fact that the continued fraction expansion of a root of a quadratic form is the reverse of its conjugate, which was first shown by E. Galois [39]. Markov's theorem in [63, 64] showed the $\mathrm{SL}(2,\mathbb{Z})$ equivalence of certain forms by their Markov minima. G. Frobenius [38] used $\mathrm{SL}(2,\mathbb{Z})$ to generate these forms.

H. Cohn's work [20, 21] showed how $\mathrm{SL}(2,\mathbb{Z})$ matrices may generate Markov numbers, equivalent to Markov forms.

Analogous to the study of binary forms is the study of ternary forms. H. Davenport studied ternary forms in [29, 30, 31, 32], as did L. J. Mordell in [69, 70]. Other work on forms include [47, 48, 72].

2.5.1 Markov spectrum and forms

Recall the definition of the Markov spectrum $\mathcal{M}$ from Section 1.1. Recall that the Markov value of a form f is

$$\frac{\sqrt{\Delta(f)}}{m(f)}.$$

Example 25. Let f be the form $f(x,y)=5x^2+11xy-5y^2$. Then the Markov value of f is

$$\frac{\sqrt{221}}{5}\in\mathcal{M}.$$

Let g be the form $g(x,y)=x^2+xy-y^2$. Then the Markov value of g is

$$\sqrt{5}\in\mathcal{M}.$$

By Theorem 1.3.19 (or alternatively Corollary 2.3.15) we have the following definition of the Markov spectrum.

Definition 2.5.1. Let $\mathbb{Z}^0$ be the set of doubly infinite sequences of positive integers $\alpha=(a_i)_{i\in\mathbb{Z}}$. Let P be a function on $\mathbb{Z}^0$ defined by the equation

$$P(\alpha)=\sup_{i\in\mathbb{Z}}\Big\{[a_i;a_{i+1}:\ldots]+[0;a_{i-1}:a_{i-2}:\ldots]\Big\}.$$

Then Markov spectrum is the set

$$\mathcal{M}=\Big\{P(\alpha)\Big|\alpha\in\mathbb{Z}^0\Big\}.$$

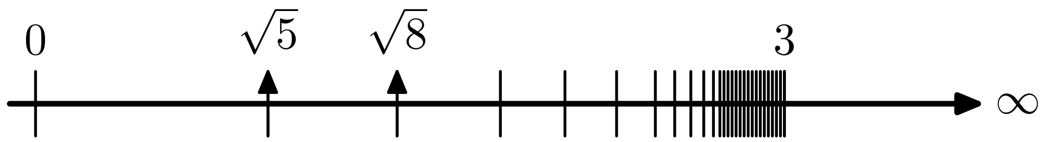

Figure 2.15: Structure of the Markov spectrum below 3.

Example 26. Let α be the doubly infinite sequence with period $(2, 2, 1, 1)$. Then

$$\begin{aligned} P(\alpha) &= [\langle 2, 2, 1, 1\rangle] + [0; \langle 1, 1, 2, 2\rangle] \\ &= \frac{\sqrt{221}+9}{10} + \frac{\sqrt{221}-9}{10} = \frac{\sqrt{221}}{5} \in \mathcal{M}. \end{aligned}$$

Let $\beta = \ldots, 1, 1, 1, \ldots$. Then

$$\begin{aligned} P(\beta) &= [1; 1 : 1 : \ldots] + [0; 1 : 1 : 1 : \ldots] \\ &= \frac{\sqrt{5}+1}{2} + \frac{\sqrt{5}-1}{2} = \sqrt{5} \in \mathcal{M}. \end{aligned}$$

The following theorem tells us about certain important values on the Markov spectrum.

Theorem 2.5.2.

(i) *The smallest value on the Markov spectrum is* $\sqrt{5}$.

(ii) *The second smallest value on the Markov spectrum is* $\sqrt{8}$.

(iii) *The first limiting point of the Markov spectrum is 3.*

The result in Part (i) of Theorem 2.5.2 is attributed to A. Hurwitz [46] (known as Hurwitz's theorem), but the result was proven earlier by A. Markov in [63, 64], as was Part (ii). In fact, both (i) and (ii) were known earlier still by Korkine and Zolotarev [59, pg. 369-370]. Part (iii) is part of Markov's theorem as stated in Section 1.1, given in the papers by A. Markov [63, 64].

Example 27. Figure 2.15 shows the structure of the Markov spectrum below 3.

Recall the definition of Markov forms $f_{Q_{a,b}}$ defined for a Markov triple (a, Q, b) from Definition 1.1.6.

Example 28. Table 2.1 shows triples where the first eight regular Markov numbers are the largest elements, the Markov forms for these triples, and the associated Markov values.

The following proposition shows when a Markov form is defined solely by its Markov number. We delay the proof until Section 3.2

Proposition 2.5.3. *Markov forms are defined uniquely by the value* $f_{Q_{a,b}}(1, 0)$ *if and only if the uniqueness conjecture for Markov numbers is true.*

Table 2.1: Markov values and Markov forms

(a, Q, b)	$\sqrt{9Q^2 - 4}/Q$	$f_{Q_{a,b}}$
$(1, 1, 1)$	$\sqrt{5}$	$x^2 + xy - y^2$
$(1, 2, 1)$	$\sqrt{8}$	$2x^2 + 4xy - 2y^2$
$(1, 5, 2)$	$\frac{\sqrt{221}}{5}$	$5x^2 + 11xy - 5y^2$
$(1, 13, 5)$	$\frac{\sqrt{1517}}{13}$	$13x^2 + 29xy - 13y^2$
$(5, 29, 2)$	$\frac{\sqrt{7565}}{29}$	$29x^2 63 + xy - 31y^2$
$(1, 34, 13)$	$\frac{\sqrt{2600}}{17}$	$34x^2 + 76xy - 34y^2$
$(1, 89, 34)$	$\frac{\sqrt{71285}}{89}$	$89x^2 + 199xy - 89y^2$
$(29, 169, 2)$	$\frac{\sqrt{257045}}{169}$	$169x^2 + 367xy - 181y^2$

2.5.2 Cohn matrices

In regular Markov theory there are collections of matrices called *Cohn matrices*, studied by H. Cohn in [20]. These are important for two reasons: firstly they mirror the graph structure of regular Markov numbers with the operation of matrix multiplication. Secondly, each Cohn matrix represents a Markov number, derived from the trace of the matrix. An exposition on Cohn matrices is found in [2], from where we get the following definition.

Definition 2.5.4. An $\mathrm{SL}(2, \mathbb{Z})$ matrix

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a *Cohn matrix* if there is a Markov number m such that

$$b = m \quad \text{and} \quad \mathrm{tr}(C) = 3m.$$

Example 29. The following are Cohn matrices.

$$A = \begin{pmatrix} 7 & 5 \\ 11 & 8 \end{pmatrix}, B = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}, X = \begin{pmatrix} 47 & 34 \\ 76 & 55 \end{pmatrix}, \text{and } Y = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Note how the multiplication of A and B results in another Cohn matrix C,

$$C = AB = \begin{pmatrix} 41 & 29 \\ 65 & 46 \end{pmatrix},$$

and that the regular Markov numbers represented by (A, C, B) form a vertex in the tree of regular Markov numbers $\mathcal{G}_\Sigma(1, 5, 2)$ from Definition 1.1.11. The vertex in the tree of regular Markov numbers is then $(5, 29, 2)$.

Cohn matrices have a triple graph structure. In fact, there are multiple different trees of Cohn matrices, all of which correspond to the tree of Markov numbers through the map taking a matrix to a third of its trace. To verify this we first need the following definitions.

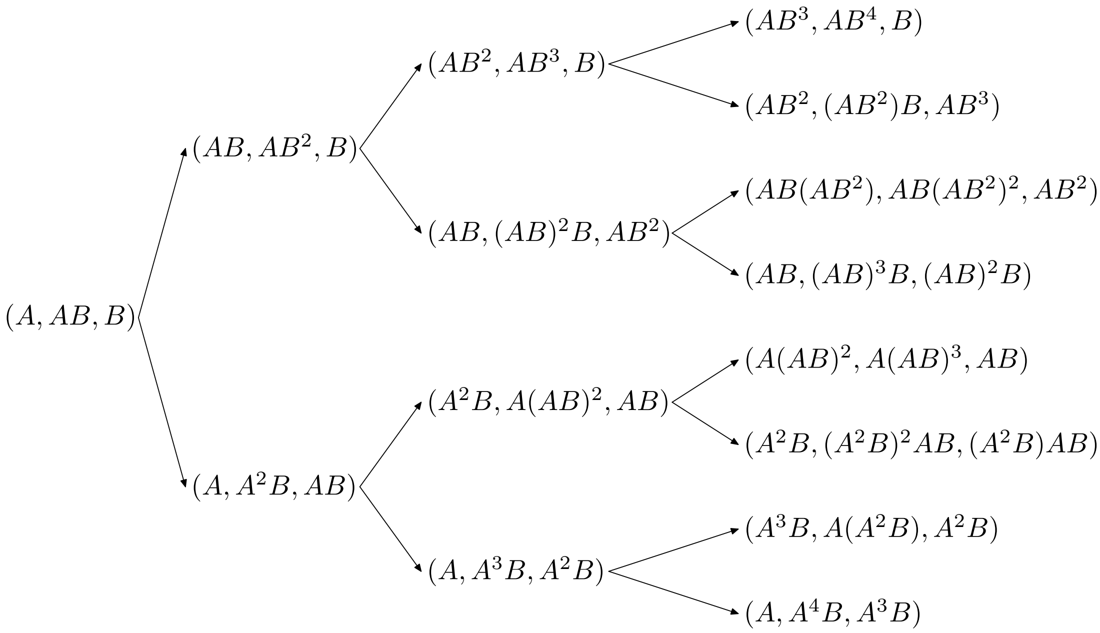

Figure 2.16: The structure of the first 4 levels in the tree of Cohn matrices with starting pair (A, B).

Definition 2.5.5.

(i) For an integer a define the pair of matrices $C_1(a)$ and $C_2(a)$ by

$$C_1(a) = \begin{pmatrix} a & 1 \\ 3a-a^2-1 & 3-a \end{pmatrix} \quad \text{and} \quad C_2(a) = \begin{pmatrix} 2a+1 & 2 \\ 4a-2a^2+2 & 5-2a \end{pmatrix}.$$

We call the pair

$$\big(C_1(a), C_2(a)\big)$$

a *starting pair of Cohn matrices for* a.

(ii) For an integer a we define $\mathcal{G}_C(a)$ to be a directed tree of $\mathrm{SL}(2,\mathbb{Z})$ matrices with root

$$\big(C_1(a), C_1(a)C_2(a), C_2(a)\big),$$

where $\big(C_1(a), C_2(a)\big)$ is the starting pair of Cohn matrices for a. Each vertex $v = (A, B, C)$ in $\mathcal{G}_C(a)$ is connected to two other vertices

$$w_L = (A, AB, B) \quad \text{and} \quad w_R = (B, BC, C)$$

by the edges (v, w_L) and (v, w_R). We call such a graph $\mathcal{G}_C(a)$ the *tree of Cohn matrices for* a.

Example 30. In Figure 2.16 we show the structure of the first four levels in a tree of Cohn matrices. Here we may substitute any starting pair of Cohn matrices for A and B.

Example 31. For the matrices

$$A = \begin{pmatrix} 18 & 13 \\ 29 & 21 \end{pmatrix}, \; C = \begin{pmatrix} 269 & 194 \\ 434 & 313 \end{pmatrix}, \; B = \begin{pmatrix} 7 & 5 \\ 11 & 8 \end{pmatrix},$$

the triple (A, C, B) is at a vertex in the graph $\mathcal{G}_C\big(1\big)$. Note that $AB = C$. Taking the trace of each matrix and dividing by 3 gives the triple

$$\left(\frac{(\operatorname{tr} A)}{3}, \frac{(\operatorname{tr} C)}{3}, \frac{(\operatorname{tr} B)}{3} \right) = (13, 194, 5),$$

which is at a vertex in the tree of regular Markov numbers.

We finish this section with a theorem that proves a general version of this example. The proof may be found in the book by M. Aigner [2].

Proposition 2.5.6.

(i) Each vertex in each tree of Cohn matrices is a triple of Cohn matrices.

(iii) Let $v = (A, B, C)$, $v_L = (A, AB, B)$, and $v_R = (B, BC, C)$ be vertices in any Cohn tree connected by the edges (v, v_L) and (v, v_R). Then the triples

$$V = \left(\frac{(\operatorname{tr} A)}{3}, \frac{(\operatorname{tr} B)}{3}, \frac{(\operatorname{tr} C)}{3} \right),$$
$$V_L = \left(\frac{(\operatorname{tr} A)}{3}, \frac{(\operatorname{tr} AB)}{3}, \frac{(\operatorname{tr} B)}{3} \right),$$
$$V_R = \left(\frac{(\operatorname{tr} B)}{3}, \frac{(\operatorname{tr} BC)}{3}, \frac{(\operatorname{tr} C)}{3} \right),$$

are vertices in the tree of Markov numbers connected by the edges (V, V_L) and (V, V_R).

Chapter 3

Triple graph structure of Markov numbers

In this chapter we study the relations between all the different areas in the theory of Markov numbers. In particular we look at forms, matrices, sequences, regular Markov numbers, and the Markov spectrum.

We start in Section 3.1 by defining general notation for the triple graph structure that is used throughout our study. We immediately apply the notation to regular Markov numbers.

In order to extend our notion of Markov numbers we require simple relations between forms, $\mathrm{SL}(2,\mathbb{Z})$ matrices, and regular Markov numbers. Cohn matrices are representatives of equivalence classes of $\mathrm{SL}(2,\mathbb{Z})$ matrices, as Markov forms are representatives of $\mathrm{SL}(2,\mathbb{Z})$ equivalence classes of forms. In Section 3.2 we choose new representatives from these classes. We show that they have a triple graph structure similar to regular Markov numbers, and we define maps between these graphs.

In Section 3.3 we present inverse maps to the graph relations from the previous sections.

We complete our study of the relations between the various graphs of Markov numbers, forms, matrices, and sequences in Section 3.4. We present a diagram relating all graphs and the Markov spectrum below 3.

3.1 Triple graph structure

In this section we define a notation for triple graphs. This is based on the tree structure of regular Markov numbers from Subsection 1.1.2.

Definition 3.1.1. Let S be an arbitrary set and let $\sigma : S^3 \to S$ be a ternary operation. For $(A, B, C) \in S^3$ define the operations $\mathcal{L}_\sigma$ and $\mathcal{R}_\sigma$ by

$$\mathcal{L}_\sigma(A, B, C) = (A, \sigma(A, B, C), B)$$
$$\mathcal{R}_\sigma(A, B, C) = (B, \sigma(B, C, A), C).$$

We define $\mathcal{G}(S, \sigma, V)$ to be the directed graph with root V, whose vertices are elements in S^3, and whose vertices v, $w \in S^3$ are connected by an edge (v, w) if either

$$w = \mathcal{L}_\sigma(v) \quad \text{or} \quad w = \mathcal{R}_\sigma(v).$$

We call σ the *triple graph operation* of $\mathcal{G}(S, \sigma, V)$.

We define a triple operation for integers.

Definition 3.1.2. Define the triple operation of integers $\Sigma : \mathbb{Z}^3 \to \mathbb{Z}$ by the relation

$$\Sigma(a, b, c) = 3ab - c.$$

The operation Σ is called *regular Markov multiplication of integer triples.*

Remark 3.1.3. Note that

$$\mathcal{G}\big(\mathbb{Z}, \Sigma, (1, 5, 2)\big) = \mathcal{G}_\Sigma(1, 5, 2)$$

is the tree of regular Markov numbers from Definition 1.1.11. This structure does not depend on the information at the vertices (in this case the triples of integers). As such, the graph of Definition 3.1.1 is also a tree.

Here and below, when we talk of triple graphs it's understood that these are in fact trees. We call $G\big(\mathbb{Z}, \Sigma, (1, 5, 2)\big)$ the *graph of regular Markov numbers.* In literature the term *Markov numbers* is common (not *regular* Markov numbers). We add the prefix to avoid confusion later when defining *general* Markov numbers.

We relate vertices of different triple graphs by looking at their paths from the roots of the graphs.

Definition 3.1.4. Let w be a vertex in $\mathcal{G}(S, \sigma, v)$. Let $a_2, \ldots, a_{2n-1}$ be positive integers, and a_1 and a_{2n} be non negative integers. We say that $(a_1, \ldots, a_{2n})$ is a *path from v to w* if

$$w = \mathcal{L}_\sigma^{a_{2n}} \mathcal{R}_\sigma^{a_{2n-1}} \ldots \mathcal{L}_\sigma^{a_2} \mathcal{R}_\sigma^{a_1}(v).$$

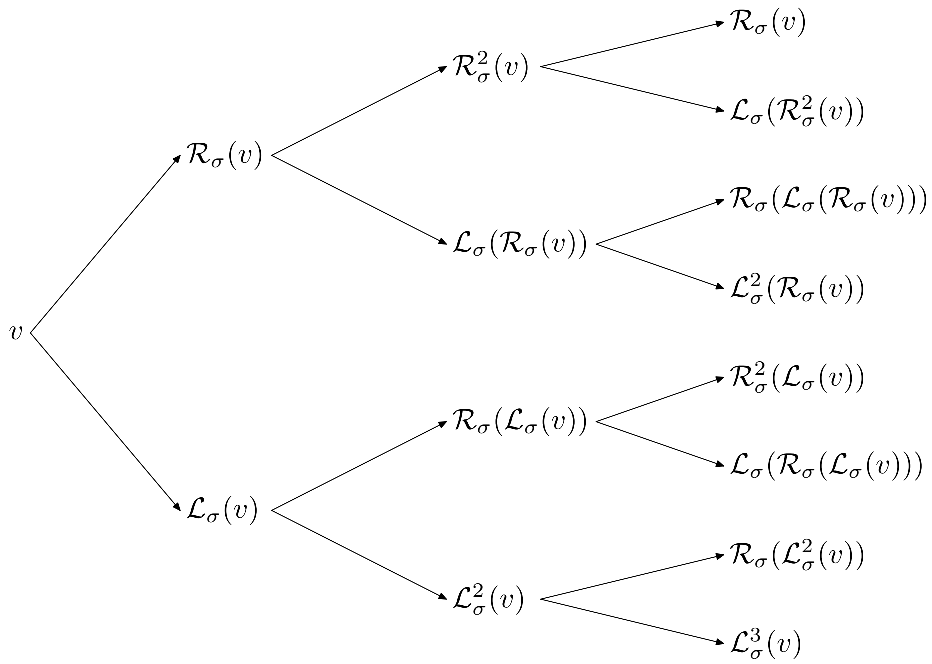

Figure 3.1: The first 4 levels in an arbitrary graph $\mathcal{G}(S, \sigma, v)$.

The power a_i indicates an operation repeated a_i times. We denote the path from v to some vertex w by $P_v(w)$. For two triple graphs and vertices $w_1 \in \mathcal{G}(S_1, \sigma_1, v_1)$ and $w_2 \in \mathcal{G}(S_2, \sigma_2, v_2)$ we say that w_1 and w_2 are *in the same position* if

$$P_{v_1}(w_1) = P_{v_2}(w_2).$$

Definition 3.1.5. We define the *N-th level* in $\mathcal{G}(S, \sigma, v)$ to be all vertices w such that the path $(a_1, \ldots, a_{2n})$ from v to w satisfies

$$\sum_{i=1}^{2n} a_i = N.$$

The 0-th level contains only the vertex v.

With these three definitions we can relate different graphs of matrices, numbers, forms, and sequences.

Example 32. Figure 3.1 shows the first four levels of a graph $\mathcal{G}(S, \sigma, v)$.

Recall the graph of of Cohn matrices $\mathcal{G}_C(a)$ for an integer a.

Definition 3.1.6. Define a triple operation of matrices $\bullet : \big(\mathrm{SL}(2,\mathbb{Z})\big)^3 \to \mathrm{SL}(2,\mathbb{Z})$ by the relation

$$\bullet(A, B, C) = AB.$$

We finish this section with the following change of notation for graphs.

Remark 3.1.7. Let

$$\mathcal{G}_C(a) = \mathcal{G}\Big(\mathrm{SL}(2,\mathbb{Z}), \bullet, \big(C_1(a), C_1(a)C_2(a), C_2(a)\big)\Big).$$

3.2 Ternary operations

In this section we define the ternary operations needed to generate the triple graphs for forms, matrices, and sequences.

We start by defining *the graph of regular Markov sequences* in Subsection 3.2.1. We discuss some general properties for these sequences, and present an important proposition justifying their definition.

In Subsection 3.2.2 we define *the graph of regular reduced Markov matrices*, based on the graph of regular Markov sequences. We define other graphs of matrices from a ternary operation and triples of $\mathrm{SL}(2,\mathbb{Z})$ matrices. We show that for a specific triple of matrices, the ternary operation also defines the graph of regular reduced Markov matrices.

In Subsection 3.2.3 we define *the graph of regular reduced Markov forms* based on the graph of regular Markov sequences. We define other graphs of forms from a ternary operation and triples of indefinite binary quadratic forms. We show that for a specific triple of forms, the ternary operation also defines the graph of regular reduced Markov forms.

We finish the section with Subsection 3.2.4. Here we show that the previously defined regular reduced Markov forms are equivalent to Markov forms, and that the regular reduced Markov matrices are similar to certain Cohn matrices.

3.2.1 Triple operation for regular Markov sequences

In this subsection we build the graph of regular Markov sequences, and show their relation to regular Markov numbers. Recall the definition of concatenation of two finite sequences.

Definition 3.2.1. Let $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_m)$ be finite sequences. The *concatenation of* α *and* β, written $\alpha \oplus \beta$ and shortened where appropriate to $\alpha\beta$, is

$$\alpha \oplus \beta = \alpha\beta = (a_1, \ldots, a_n, b_1, \ldots, b_m).$$

We define a graph of sequences of positive integers.

Definition 3.2.2. Let the set of finite sequences of integers be denoted by $\mathbb{Z}^\infty$. Let (α, β, δ) be a triple of sequences of positive integers in $\big(\mathbb{Z}^\infty\big)^3$ such that $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_m)$. Define the operation $\widehat{\oplus}$ by

$$\widehat{\oplus}(\alpha, \beta, \delta) = \alpha \oplus \beta.$$

Then

$$\mathcal{G}\big(\mathbb{Z}^\infty, \widehat{\oplus}, (\alpha, \alpha\beta, \beta)\big)$$

is the *graph of sequences for* α *and* β. We denote this graph by $\mathcal{G}_{\oplus}(\alpha, \beta)$.

We call the graph $\mathcal{G}_{\oplus}\big((1,1),(2,2)\big)$ the *graph of regular Markov sequences.* The sequences in the vertices of this graph are called *regular Markov sequences.*

Example 33. In Figure 3.2 we show first three levels in the graph of regular Markov sequences.

Remark 3.2.3. For a graph $\mathcal{G}\big(S, \sigma, (s_1, s_2, s_3)\big)$, if σ may be reduced to a binary operation i.e. there is some $\widehat{\sigma} : S^2 \to S$ such that for any $(a, b, c) \in S^3$ we have

$$\sigma(a, b, c) = \widehat{\sigma}(a, b),$$

then we may denote the graph $\mathcal{G}\big(S, \sigma, (s_1, s_2, s_3)\big)$ by

$$\mathcal{G}_{\sigma}(s_1, s_3).$$

Otherwise we denote the graph by

$$\mathcal{G}_{\sigma}(s_1, s_2, s_3).$$

We note several facts about regular Markov sequences in the following remark and proposition.

Remark 3.2.4. For every regular Markov sequence $\alpha = (a_1, \ldots, a_n)$ we have, by construction, that n is an even integer. Every α other than $(1, 1)$ and $(2, 2)$ begins with the subsequence $(1, 1)$ and ends with the subsequence $(2, 2)$.

Proposition 3.2.5. *For a regular Markov sequence* $\alpha = (a_1, \ldots, a_n)$ *the subsequence* $(a_3, \ldots, a_{n-2})$ *is symmetric.*

A proof of this proposition may be found in the book by T. Cusick and M. Flahive [23, Theorem 3].

Example 34. The sequence $\alpha = (1, 1, 2, 2, 1, 1, 2, 2, 2, 2)$ is a regular Markov sequence. Note that the length of the sequence is even, and that the subsequence

$$(2, 2, 1, 1, 2, 2)$$

is palindromic.

Example 35. Table 3.1 shows some regular Markov sequences in the left hand column and some sequences that do not appear in vertices of $\mathcal{G}_{\oplus}\big((1,1),(2,2)\big)$ in the right hand column.

We relate the graph of regular Markov sequences with the graph of regular Markov numbers.

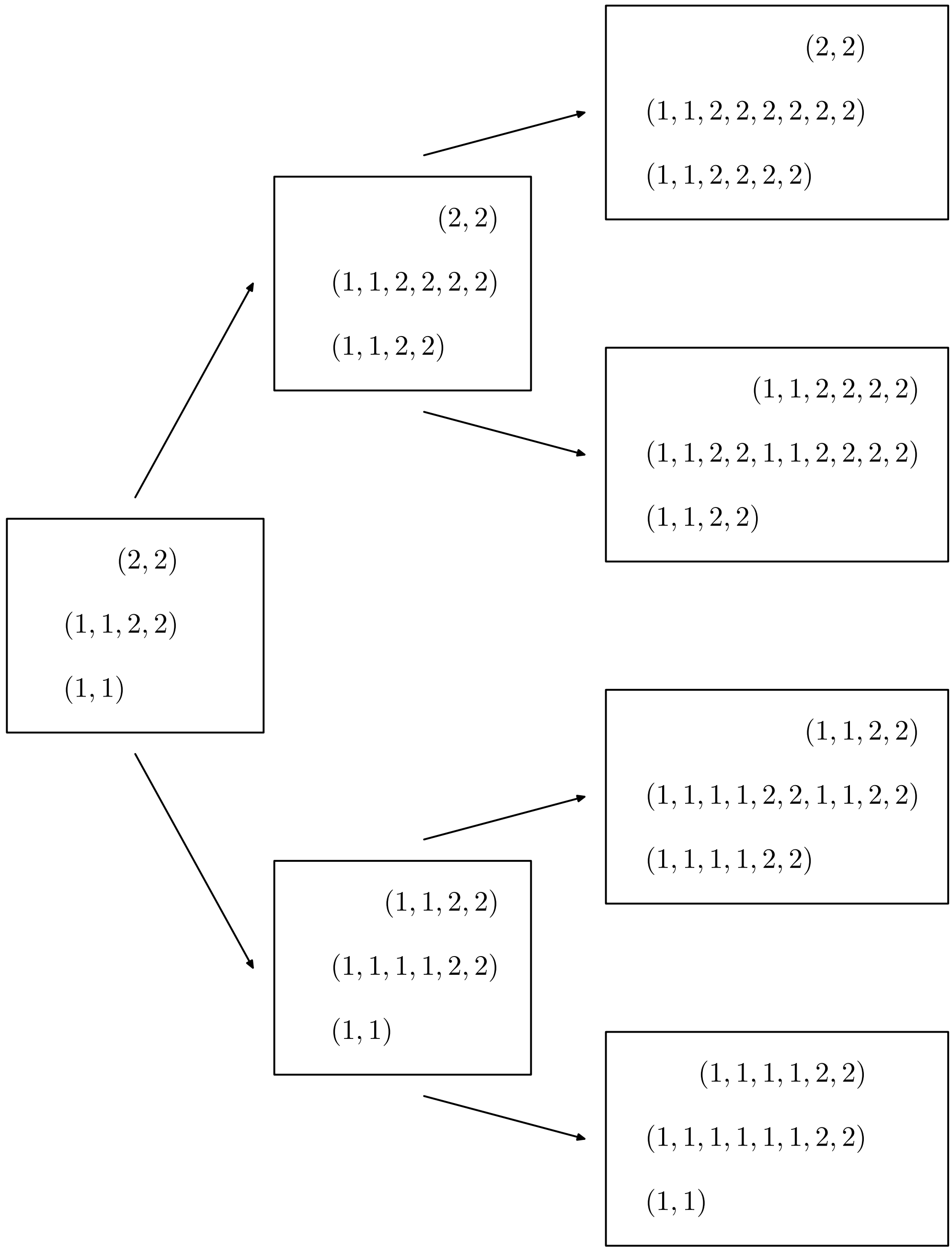

Figure 3.2: The first three levels of $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$.

Table 3.1: Some sequences in $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$ and some sequences not in $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$.

Sequences in $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$	Sequences not in $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$
$(1,1)$, $(1,1,2,2)$, $(1,1,2,2,1,1,2,2,2,2)$, $(1,1,2,2,1,1,2,2,2,2,1,1,2,2,2,2)$.	$(1,2,2,1)$, $(1,1,2,2,1,1,3,2,2,2)$, $(1,1,2,2,2,2,2)$, $(1,1,1,2,1,1,1)$.

$$\begin{tikzcd}
 & & \big(\mu\nu^2, \mu\nu^3, \nu\big) \\
 & \big(\mu\nu, \mu\nu^2, \nu\big) \arrow[ur] \arrow[dr] & \\
 & & \big(\mu\nu, (\mu\nu)^2\nu, \mu\nu^2\big) \\
\big(\mu, \mu\nu, \nu\big) \arrow[uur] \arrow[ddr] & & \\
 & & \big(\mu^2\nu, \mu(\mu\nu)^2, \mu\nu\big) \\
 & \big(\mu, \mu^2\nu, \mu\nu\big) \arrow[ur] \arrow[dr] & \\
 & & \big(\mu, \mu^3\nu, \mu^2\nu\big)
\end{tikzcd}$$

Figure 3.3: The first three levels in a graph $\mathcal{G}_{\oplus}(\mu, \nu)$.

$$\begin{tikzcd}
 & & \Big(\breve{K}(\mu\nu^2), \breve{K}(\mu\nu^3), \breve{K}(\nu)\Big) \\
 & \Big(\breve{K}(\mu\nu), \breve{K}(\mu\nu^2), \breve{K}(\nu)\Big) \arrow[ur] \arrow[dr] & \\
 & & \Big(\breve{K}(\mu\nu), \breve{K}\big((\mu\nu)^2\nu\big), \breve{K}(\mu\nu^2)\Big) \\
\Big(\breve{K}(\mu), \breve{K}(\mu\nu), \breve{K}(\nu)\Big) \arrow[uur] \arrow[ddr] & & \\
 & & \Big(\breve{K}(\mu^2\nu), \breve{K}\big(\mu(\mu\nu)^2\big), \breve{K}(\mu\nu)\Big) \\
 & \Big(\breve{K}(\mu), \breve{K}(\mu^2\nu), \breve{K}(\mu\nu)\Big) \arrow[ur] \arrow[dr] & \\
 & & \Big(\breve{K}(\mu), \breve{K}(\mu^3\nu), \breve{K}(\mu^2\nu)\Big)
\end{tikzcd}$$

Figure 3.4: The first three levels in a graph $X\big(\mathcal{G}_{\oplus}(\mu, \nu)\big)$.

Definition 3.2.6. Define a map χ between triples of sequences and triples of integers by

$$\chi(\alpha, \delta, \beta) = \Big(\breve{K}(\alpha), \breve{K}(\delta), \breve{K}(\beta)\Big).$$

Define a map X taking $\mathcal{G}_{\oplus}(\mu, \nu)$ to a triple graph of integers in the following way: let vertices defined by triples v in $\mathcal{G}_{\oplus}(\mu, \nu)$ be mapped to vertices with triples $\chi(v)$, and let edges defined by the triples (v, w) be mapped to edges defined by the triples $\big(\chi(v), \chi(w)\big)$. Denote this new graph by $X\big(\mathcal{G}_{\oplus}(\mu, \nu)\big)$.

Example 36. Figures 3.3 and 3.4 compare the first three levels in a graph $\mathcal{G}_{\oplus}(\mu, \nu)$ with $X\big(\mathcal{G}_{\oplus}(\mu, \nu)\big)$.

We have a proposition relating the graph of regular Markov sequences to the graph of regular Markov numbers.

Proposition 3.2.7. *We have the following equality.*

$$X\Big(\mathcal{G}_{\oplus}\big((1,1),(2,2)\big)\Big) = \mathcal{G}_{\Sigma}(1, 5, 2).$$

Example 37. Table 3.2 shows some regular Markov sequences α in the left hand column and their regular Markov numbers $\breve{K}(\alpha)$ in the right hand column.

Proposition 3.2.7 has been known at least since G. Frobenius [38]. It is a corollary to the work of H. Cohn [20, 21], also seen in [2, Thm. 4.7]. We prove this theorem separately on page 79.

Table 3.2: Some regular Markov sequences and their regular Markov numbers.

Regular Markov sequences α,	$\breve{K}(\alpha)$
$(1,1)$	1
$(2,2)$	2
$(1,1,2,2)$	5
$(1,1,1,1,2,2)$	13
$(1,1,2,2,2,2)$	29
$(1,1,1,1,1,1,2,2)$	34
$(1,1,2,2,2,2,2,2)$	169
$(1,1,1,1,2,2,1,1,2,2)$	194
$(1,1,2,2,1,1,2,2,2,2)$	433

Proposition 3.2.8. *The graph $\mathcal{G}_{\oplus}(\mu,\nu)$ is a tree.*

We omit the proof to this proposition here. It may be seen for example in [2, 55].

We now prove Proposition 2.5.3 from Section 2.5, which says that Markov forms are defined uniquely by their regular Markov number if and only if the uniqueness conjecture for regular Markov numbers is true.

Proof of Proposition 2.5.3. Let (a, Q, b) be a Markov triple, and let u and v be integers defined as in Theorem 1.1.9 such that the Markov form of $Q_{a,b}$ is

$$f_{Q_{a,b}}(x,y) = Qx^2 + (3Q-2u)xy + (v-3u)y^2.$$

Assume Q defines the form $f_{Q_{a,b}}$ uniquely. Then Q defines u, and hence v, uniquely. Then from the odd continued fraction

$$\frac{Q}{u} = [y_{n-1},\dots,y_1]$$

we obtain the sequence $\gamma = (y_1,\dots,y_{n-1},2)$. The LLS sequence of the form has period γ, since the value $u = K_1^{n-2}(\gamma)$ is given by the LLS sequence, see [23, Theorem 3]. For two Markov triples (a_1, Q, b_1) and (a_2, Q, b_2), we have corresponding vertices $(\alpha_1,\gamma,\beta_1)$ and $(\alpha_2,\gamma,\beta_2)$ in the tree $\mathcal{G}_{\oplus}\big((1,1),(2,2)\big)$. Since the splitting of regular Markov sequences in triples is unique (see for example [9, 55]) we have that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, from which we have that $a_1 = a_2$ and $b_1 = b_2$.

If the uniqueness conjecture holds then each Markov number Q defines its vertex (a, Q, b) in the graph of Markov numbers, and so Q immediately defines the Markov form $f_{Q_{a,b}}$. ■

Remark 3.2.9. Note that the same requirement does not hold for regular Markov sequences. There could still be distinct regular Markov sequences α and β for which $\breve{K}(\alpha) = \breve{K}(\beta)$.

3.2.2 Regular reduced Markov Matrices

We define new matrices to replace Cohn matrices. We will give two definitions, one from sequences and another from a ternary operation. We show that the definitions are equivalent in Theorem 3.2.20. The advantage of the new definitions is to have simple relations between the matrices, regular Markov sequences, and forms.

We start by relating matrices to arrangements, and see how we may recover the LLS sequence for an arrangement from its matrix.

Definition 3.2.10. Let M be the matrix with integer elements and positive real eigenvalues

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with determinant $|M|$, and with $(a+d)^2 > 4|M|$. Let

$$f_M(x, y) = cx^2 + (d-a)xy - by^2.$$

We call f_M the *form associated with M*.

Proposition 3.2.11. *Let M be a matrix as described above and let f_M be the form associated with M. Then the two eigenvectors of M create an arrangement given by the equation*

$$f_M(x, y) = 0.$$

Proof. Let τ denote an eigenvalue of M. Then the eigenvectors are solutions to the equation

$$M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \tau \begin{pmatrix} x \\ y \end{pmatrix}.$$

From this we get the simultaneous equations

$$\tau = \frac{ax+by}{x} \qquad \text{and} \qquad \tau = \frac{cx+dy}{y},$$

and so

$$axy + by^2 = cx^2 + dxy.$$

Hence the eigenvectors are defined by the equation $f(x, y) = 0$, where

$$f(x, y) = cx^2 + (d-a)xy - by^2.$$

By the condition that $(a+d)^2 > 4|M|$, we have that the discriminant

$$\Delta(f) = (d-a)^2 + 4bc$$

is positive. ■

Definition 3.2.12. The *arrangement* of a 2×2 integer matrix A with positive eigenvalues is the pair of lines defined by the eigenvectors of A. The *LLS sequence* associated with A is the LLS sequence of its arrangement.

We introduce a notion of reduced matrices. This is important as the LLS sequence of a reduced matrix is easily computable, see Theorem 3.2.15 below.

Definition 3.2.13. We call a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

reduced if $d > c \geq a \geq 0$.

Remark 3.2.14. There exists an algorithm to quickly compute the LLS sequence of a reduced matrix, see the book by O. Karpenkov [52, pg. 77]. There may be multiple reduced matrices for any given arrangement. We classify these for periodic sequences in Subsection 4.3.2 below. For further information see [52, Section 7.2]

The following theorem by O. Karpenkov allows us to extract the LLS sequence of an arrangement from its matrix. A proof may be found in [52].

Theorem 3.2.15 (O. Karpenkov [52])**.** *Let M be a reduced matrix in* $\mathrm{SL}(2,\mathbb{Z})$ *given by*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Assuming $a > 0$, let the odd regular continued fraction expansion of $\frac{c}{a}$ be

$$[a_1 : \ldots : a_{2n-1}]$$

and let $a_{2n} = \lfloor \frac{d-1}{c} \rfloor$. Then the LLS sequence of the arrangement given by M is periodic, with period

$$(a_1, \ldots, a_{2n}).$$

If $a = 0$ then the LLS sequence for M is periodic with period

$$(1, d-2).$$

Matrices from sequences

We develop a definition for our new matrices from finite sequences.

Recall from the proof of Proposition 2.2.22 that $[\langle\alpha\rangle]$ is a solution to the quadratic equation

$$K_1^{n-1}(\alpha) - \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)y - K_2^n(\alpha)y^2 = 0,$$

and $\big(-1, [\langle\alpha\rangle]\big)$ is a solution to $f(x,y) = 0$, where

$$f(x,y) = K_1^{n-1}(\alpha)x^2 + \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)xy - K_2^n(\alpha)y^2.$$

Combining this fact with Proposition 3.2.11 we have the following definition.

Definition 3.2.16. Let $\alpha = (a_1, \dots, a_n)$ be a sequence of positive integers with $n > 1$. Define the matrix

$$M_\alpha = \begin{pmatrix} K_2^{n-1}(\alpha) & K_2^n(\alpha) \\ K_1^{n-1}(\alpha) & K_1^n(\alpha) \end{pmatrix}$$

to be the *reduced matrix for* α. If α is a regular Markov sequence then we call M_α the *regular reduced Markov matrix for* α.

Example 38. Here we show some regular reduced Markov matrices.

$$M_\alpha = \begin{pmatrix} 254 & 613 \\ 433 & 1045 \end{pmatrix} \qquad M_\beta = \begin{pmatrix} 99 & 239 \\ 169 & 408 \end{pmatrix} \qquad M_\delta = \begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix}$$
$$\alpha = (1,1,2,2,1,1,2,2,2,2) \qquad \beta = (1,1,2,2,2,2,2,2) \qquad \delta = (1,1,2,2)$$

Remark 3.2.17. Note that the matrices in Definition 3.2.16 are reduced. If α is $(1,1)$ or $(2,2)$ then $K_2^1(\alpha) = 1$, and otherwise $K_2^{n-1}(\alpha)$ is a product of positive integers, and so positive itself. By the definition of continuants we have that

$$K_1^n(\alpha) = a_n K_1^{n-1}(\alpha) + K_1^{n-1}(\alpha) > K_1^{n-1}(\alpha),$$
$$K_1^{n-1}(\alpha) = a_1 K_2^{n-1}(\alpha) + K_3^{n-1}(\alpha) > K_2^{n-1}(\alpha).$$

Note also that the transpose of such matrices are also reduced, since

$$K_1^n(\alpha) = a_1 K_2^n(\alpha) + K_3^n(\alpha) > K_2^n(\alpha).$$

However, Cohn matrices are not necessarily reduced in this sense.

Now we define a triple graph of regular reduced matrices from triple graphs of sequences.

Definition 3.2.18. We define a map γ between triples of sequences and triples of 2×2 matrices by

$$\gamma(\alpha, \delta, \beta) = (M_\alpha, M_\delta, M_\beta).$$

Define a map Γ taking a graph $\mathcal{G}_\oplus(\mu, \nu)$ to a triple graph of matrices in the following way: let vertices defined by triples v in $\mathcal{G}_\oplus(\mu, \nu)$ be mapped to vertices with triples $\gamma(v)$, and let edges defined by the triples (v, w) be mapped to edges defined by the triples $\big(\gamma(v), \gamma(w)\big)$. Denote the new graph by $\Gamma\big(\mathcal{G}_\oplus(\mu, \nu)\big)$. We call $\Gamma\big(\mathcal{G}_\oplus\big((1,1),(2,2)\big)\big)$ the *graph of regular reduced Markov matrices.*

Matrices from ternary operations

Now we define matrices from a ternary operation.

Definition 3.2.19. Recall the notation

$$\mathcal{G}\Big(\,\mathrm{SL}(2,\mathbb{Z}), \bullet, \big(C_1(a), C_1(a)C_2(a), C_2(a)\big)\Big)$$

for the graph of Cohn matrices for some positive integer a. For sequences α and β of positive integers with length greater than one, and the same ternary operation $\bullet$, we define the graph

$$\mathcal{G}\big(\,\mathrm{SL}(2,\mathbb{Z}), \bullet, (M_\alpha, M_\alpha M_\beta, M_\beta)\,\big),$$

where M_α and M_β are the regular reduced matrices for the sequences α and β. We denote this graph $\mathcal{G}_\bullet(M_\alpha, M_\beta)$.

Matrix definition equivalence

We show that the two previous definitions for graphs of matrices are equivalent for regular Markov sequences.

Proposition 3.2.20. *We have the following equality:*

$$\Gamma\big(\mathcal{G}_\oplus\big((1,1),(2,2)\big)\big) = \mathcal{G}_\bullet\big(M_{(1,1)}, M_{(2,2)}\big).$$

First we prove the following important lemma on the multiplication of regular reduced matrices defined by sequences.

Lemma 3.2.21. Let $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_m)$ be finite sequences of positive integers with $n \geq 2$ and $m \geq 2$. Then

$$\begin{pmatrix} K_2^{n-1}(\alpha) & K_2^n(\alpha) \\ K_1^{n-1}(\alpha) & K_1^n(\alpha) \end{pmatrix} \begin{pmatrix} K_2^{m-1}(\beta) & K_2^m(\beta) \\ K_1^{m-1}(\beta) & K_1^m(\beta) \end{pmatrix} = \begin{pmatrix} K_2^{n+m-1}(\alpha\beta) & K_2^{n+m}(\alpha\beta) \\ K_1^{n+m-1}(\alpha\beta) & K_1^{n+m}(\alpha\beta) \end{pmatrix}$$

Remark 3.2.22. This lemma says in particular that for a triple of regular Markov sequences $(\alpha, \alpha\beta, \beta)$, the associated regular Markov matrices $(M_\alpha, M_\alpha M_\beta, M_\beta)$ obey the relation

$$M_\alpha M_\beta = M_{\alpha\beta}.$$

Proof of Lemma 3.2.21. We need the four equalities to hold

$$\begin{aligned} K_2^{n+m-1}(\alpha\beta) &= K_2^n(\alpha)K_1^{m-1}(\beta) + K_2^{n-1}(\alpha)K_2^{m-1}(\beta), \\ K_2^{n+m}(\alpha\beta) &= K_2^n(\alpha)K_1^m(\beta) + K_2^{n-1}(\alpha)K_2^m(\beta), \\ K_1^{n+m-1}(\alpha\beta) &= K_1^n(\alpha)K_1^{m-1}(\beta) + K_1^{n-1}(\alpha)K_2^{m-1}(\beta), \text{ and} \\ K_1^{n+m}(\alpha\beta) &= K_1^n(\alpha)K_1^m(\beta) + K_1^{n-1}(\alpha)K_2^m(\beta). \end{aligned}$$

Each of these equalities hold by direct application of Proposition 2.1.6. ■

We prove the earlier Proposition 3.2.20.

Proof of Proposition 3.2.20. By inspection we have

$$\gamma\big((1,1),(1,1,2,2),(2,2)\big) = \big(M_{(1,1)}, M_{(1,1)}M_{(2,2)}, M_{(2,2)}\big).$$

This is the base of induction.

Assume for a triple of sequences $(\alpha, \alpha\beta, \beta)$ at a vertex in $\mathcal{G}_{\oplus}\big((1,1),(2,2)\big)$ that $\gamma(\alpha, \alpha\beta, \beta) = \big(M_\alpha, M_\alpha M_\beta, M_\beta\big)$. We show that

$$\gamma(\alpha, \alpha\alpha\beta, \alpha\beta) = \big(M_\alpha, M_\alpha M_{\alpha\beta}, M_{\alpha\beta}\big),$$
$$\gamma(\alpha\beta, \alpha\beta\beta, \beta) = \big(M_{\alpha\beta}, M_{\alpha\beta} M_\beta, M_\beta\big).$$

Both equalities follow directly from Lemma 3.2.21. This concludes the induction, and the proof. ■

Remark 3.2.23. Note that by Lemma 3.2.21 the proof of this theorem implies that the equality of graphs

$$\Gamma\big(\mathcal{G}_{\oplus}(\mu, \nu)\big) = \mathcal{G}_{\bullet}(M_\mu, M_\nu)$$

holds for any finite sequences of positive integers μ and ν with length greater than or equal to two.

Remark 3.2.24. It is known for a regular Markov sequence $\alpha = (a_1, \ldots, a_n)$ with regular Markov number $Q = \breve{K}(\alpha)$ that

$$K_1^n(\alpha) + K_2^{n-1}(\alpha) = 3\breve{K}(\alpha) = 3Q. \tag{3.1}$$

One can find a derivation of this relation in, for example, [23]. We re-derive this in Section 4.1.

Equation (3.1) tells us that the trace of regular reduced Markov matrices is three times the associated regular Markov number, just like for Cohn matrices. One might ask then if the two sets of matrices are similar, that is, given a regular reduced Markov matrix M_α, is there an $\mathrm{SL}(2,\mathbb{Z})$ matrix P such that $P^{-1}M_\alpha P$ is a Cohn matrix?. In fact we prove that this is the case with the upcoming Proposition 3.2.39.

3.2.3 Regular reduced Markov forms

In this subsection we define triple graphs of forms. We define graphs of forms by regular Markov sequences and also ternary operations. We show that these definitions are equivalent in Proposition 3.2.31.

Forms from sequences

We define forms in a natural way from continued fractions of sequences. Recall that a periodic regular continued fraction $[\langle\alpha\rangle]$ with period $\alpha = (a_1, \ldots, a_n)$ is a solution to the quadratic equation

$$K_1^{n-1}(\alpha) + \big(K_2^{n-1}(\alpha) - K_1^n(\alpha)\big)y - K_2^n(\alpha)y^2 = 0. \tag{3.2}$$

With this in mind we define new forms based on finite sequences.

Table 3.3: Some regular reduced Markov forms.

Regular Markov sequence α	Regular reduced form f_α
$(2,2)$	$2x^2+4xy-2y^2$
$(1,1,2,2)$	$5x^2+9xy-7y^2$
$(1,1,1,1,2,2)$	$13x^2+23xy-19y^2$
$(1,1,2,2,2,2)$	$29x^2+53xy-41y^2$
$(1,1,1,1,2,2,1,1,2,2)$	$194x^2+344xy-284y^2$
$(1,1,2,2,1,1,2,2,2,2)$	$433x^2+791xy-613y^2$

Definition 3.2.25. Let $\alpha = (a_1, \ldots, a_n)$ be a sequence of positive integers, where $n > 1$. Let f_α be the form defined by the equation

$$f_\alpha(x, y) = \breve{K}(\alpha)x^2 + \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)xy - K_2^n(\alpha)y^2.$$

We call f_α the *form associated with* α. If α is a regular Markov sequence we call f_α the *regular reduced Markov form of* α.

Remark 3.2.26. Note that we choose a form $f(x, y)$ with the coefficient of xy being

$$K_1^n(\alpha) - K_2^{n-1}(\alpha),$$

which is the negative of the same coefficient in Equation (3.2). We choose this to ensure that the associated Markov matrix is reduced, in the sense of Definition 3.2.13.

Note that the two forms $f(x, y)$ and $f(-x, y)$ attain the same values over integer points and have the same discriminant, and hence their Markov value is the same. We discuss this more in Subsection 3.2.4.

Example 39. Table 3.3 shows regular reduced Markov forms f_α and their regular Markov sequences α.

Recall Proposition 2.2.34, which says that the Markov minimum of the form f_α associated with a sequence $\alpha = (a_1, \ldots, a_n)$ is

$$\breve{K}(\alpha_i),$$

for some $i = 1, \ldots, n$, where α_i is the i-th cyclic shift of α.

Remark 3.2.27. The Markov minimum for a Markov form $f_{Q_{a,b}}$ is

$$f_{Q_{a,b}}(1, 0) = Q.$$

Example 40. The regular reduced Markov form for $\alpha = (1, 1, 2, 2)$ is

$$f_\alpha(x, y) = 5x^2 + 9xy - 7y^2.$$

The values of $\breve{K}(\alpha_i)$ for circular shifts α_i of α are

$$\{5, 7\}.$$

Note that we have only two values, since $\breve{K}(\alpha) = \breve{K}(\alpha_4)$. Hence $m(f_\alpha) = f_\alpha(1, 0)$.

Let $\beta = (5, 3, 4, 3)$. Then

$$f_\beta(x, y) = 69x^2 + 210xy - 42y^2.$$

The values of $\breve{K}(\beta_i)$ for circular shifts β_i of β are

$$\{69, 42, 51\}.$$

Hence $m(f_\beta) = f_\beta(0, 1)$.

Remark 3.2.28. Note the following about the discriminant of a form f_α for some $\alpha = (a_1, \ldots, a_n)$.

$$\begin{aligned}
\Delta(f_\alpha) &= \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)^2 + 4K_1^{n-1}(\alpha)K_2^n(\alpha) \\
&= \big(K_1^n(\alpha) + K_2^{n-1}(\alpha)\big)^2 - 4\big(K_1^n(\alpha)K_2^{n-1}(\alpha) - K_1^{n-1}(\alpha)K_2^n(\alpha)\big) \\
&= \big(K_1^n(\alpha) + K_2^{n-1}(\alpha)\big)^2 - 4.
\end{aligned}$$

The third equality holds from Proposition 2.1.8 (*iii*).

We define a triple graph of regular reduced Markov forms from the graph of regular Markov sequences.

Definition 3.2.29. Define a map φ between triples of finite sequences and triples of forms by

$$\varphi(\alpha, \delta, \beta) = (f_\alpha, f_\delta, f_\beta).$$

Define a map Φ between the graph $\mathcal{G}_\oplus(\mu, \nu)$ and a triple graph of forms in the following way: let vertices defined by triples v in $\mathcal{G}_\oplus(\mu, \nu)$ be mapped to vertices with triples $\varphi(v)$, and let edges defined by the triples (v, w) be mapped to edges defined by the triples $\big(\varphi(v), \varphi(w)\big)$. Denote this graph by $\Phi\left(\mathcal{G}_\oplus(\mu, \nu)\right)$. We call $\Phi\big(\mathcal{G}_\oplus\big((1, 1), (2, 2)\big)\big)$ the *graph of regular reduced Markov forms.*

Forms from ternary operations

Now we define a triple graph of forms from a ternary operation. We show that the triple graphs of forms obtained from sequences are the same as from ternary operations.

Definition 3.2.30. Let $\mathcal{F}$ denote the set of all indefinite binary quadratic forms. Let $f_a(x, y) = a_1x^2 + a_2xy + a_3y^2$ and $f_b(x, y) = b_1x^2 + b_2xy + b_3y^2$ be in $\mathcal{F}$, and let

$$A = \sqrt{a_2^2 - 4(a_1a_3 - 1)} \quad \text{and} \quad B = \sqrt{b_2^2 - 4(b_1b_3 - 1)}.$$

Let also

$$c_1 = \frac{-a_1b_2 + a_1B + b_1a_2 + b_1A}{2},$$
$$c_2 = \frac{a_2B + b_2A}{2} - a_1b_3 + a_3b_1,$$
$$c_3 = \frac{-a_2b_3 + b_3A + a_3b_2 + a_3B}{2}.$$

Define a binary operation for forms $\widehat{\theta} : \mathcal{F}^2 \to \mathcal{F}$ by

$$\widehat{\theta}(f_a, f_b) = f_c,$$

where

$$f_c(x,y) = c_1x^2 + c_2xy + c_3y^2.$$

Define a ternary operation for forms $\theta : \mathcal{F}^3 \to \mathcal{F}$ by

$$\theta\left(f_1, f_2, f_3\right) = \widehat{\theta}\left(f_1, f_2\right).$$

We call the graph

$$\mathcal{G}\big(\mathcal{F}, \theta, \big(f, \widehat{\theta}(f,g), g\big)\big)$$

the *triple graph of forms for f and g* and denote it by $\mathcal{G}_\theta\left(f,g\right)$. We call

$$\mathcal{G}_\theta\left(f_{(1,1)}, f_{(2,2)}\right)$$

the *triple graph of regular reduced Markov forms.*

Example 41. The regular reduced Markov forms associated with the regular Markov numbers 1 and 2 are $f(x,y) = x^2 + xy - y^2$ and $g(x,y) = 2x^2 + 4xy - 2y^2$. The corresponding triple in the graph of regular Markov numbers is $(1,5,2)$. We show that the form

$$\widehat{\theta}(f,g)$$

is associated with the regular Markov number 5. For this operation we have that the constants A, B, and the coefficients c_1, c_2, and c_3 are given by

$$A = \sqrt{1 - 4(-1-1)} = 3,\qquad B = \sqrt{16 - 4(-4-1)} = 6,$$

$$c_1 = \frac{-4+6+2+6}{2} = 5,\qquad c_2 = \frac{6+12}{2} - 2 + 2 = 9,\qquad c_3 = \frac{2-6-4-6}{2} = -7.$$

Hence

$$\widehat{\theta}(f,g) = 5x^2 + 9xy - 7y^2.$$

The vertex in the graph $\Phi\Big(\mathcal{G}_\oplus\big((1,1),(2,2)\big)\Big)$ associated with

$$v = \big((1,1,2,2),(1,1,2,2,1,1,2,2,2,2),(1,1,2,2,2,2)\big)$$

is

$$\varphi(v) = \Big(5x^2 + 9xy - 7y^2, 433x^2 + 791xy - 613y^2, 29x^2 + 53xy - 41y^2\Big).$$

We have that

$$A = \sqrt{9^2 - 4(-35-1)} = 15,$$
$$B = \sqrt{53^2 - 4(-1189-1)} = 87,$$

$$c_1 = \frac{-265 + 435 + 261 + 435}{2} = 433,$$
$$c_2 = \frac{783 + 795}{2} + 205 - 203 = 791,$$
$$c_3 = \frac{369 - 615 - 371 - 609}{2} = -613.$$

Hence $\widehat{\theta}\big(5x^2 + 9xy - 7y^2, 29x^2 + 53xy - 41y^2\big) = 433x^2 + 791xy - 613y^2$.

Example 42. The Figure 3.5 shows the forms in the first 3 levels in the graph of regular reduced Markov forms.

Form definition equivalence

Now we show that the graph $\mathcal{G}_\theta\left(f_{(1,1)}, f_{(2,2)}\right)$ is obtained by replacing every sequence $\alpha = (a_1, \ldots, a_n)$ in the graph $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$ with the form

$$f_\alpha(x,y) = K_1^{n-1}(\alpha)x^2 + \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)xy - K_2^n(\alpha)y^2.$$

Proposition 3.2.31. *We have the following equality of graphs*

$$\Phi\left(\mathcal{G}_\oplus\big((1,1),(2,2)\big)\right) = \mathcal{G}_\theta\left(f_{(1,1)}, f_{(2,2)}\right).$$

Proof. First note that to get from a form $c_1x^2 + c_2xy + c_3y^2$ with c_1, $c_2 > 0$, and $c_3 < 0$ to its associated reduced matrix we need to find a number C such that

$$\begin{pmatrix} C & -c_3 \\ c_1 & C + c_2 \end{pmatrix}$$

is a reduced $\mathrm{SL}(2,\mathbb{Z})$ matrix. From the determinant of this matrix being 1 we have that C must satisfy the equation

$$C^2 + c_2C + c_1c_3 - 1 = 0.$$

From the requirement that the matrix be reduced we have that

$$C = \frac{-c_2 + \sqrt{c_2^2 - 4(c_1c_3 - 1)}}{2}.$$

From Proposition 3.2.20 we only need show that the operation of two forms

$$\widehat{\theta}(f_\alpha, f_\beta)$$

results in the form associated with the matrix

$$M_\alpha M_\beta,$$

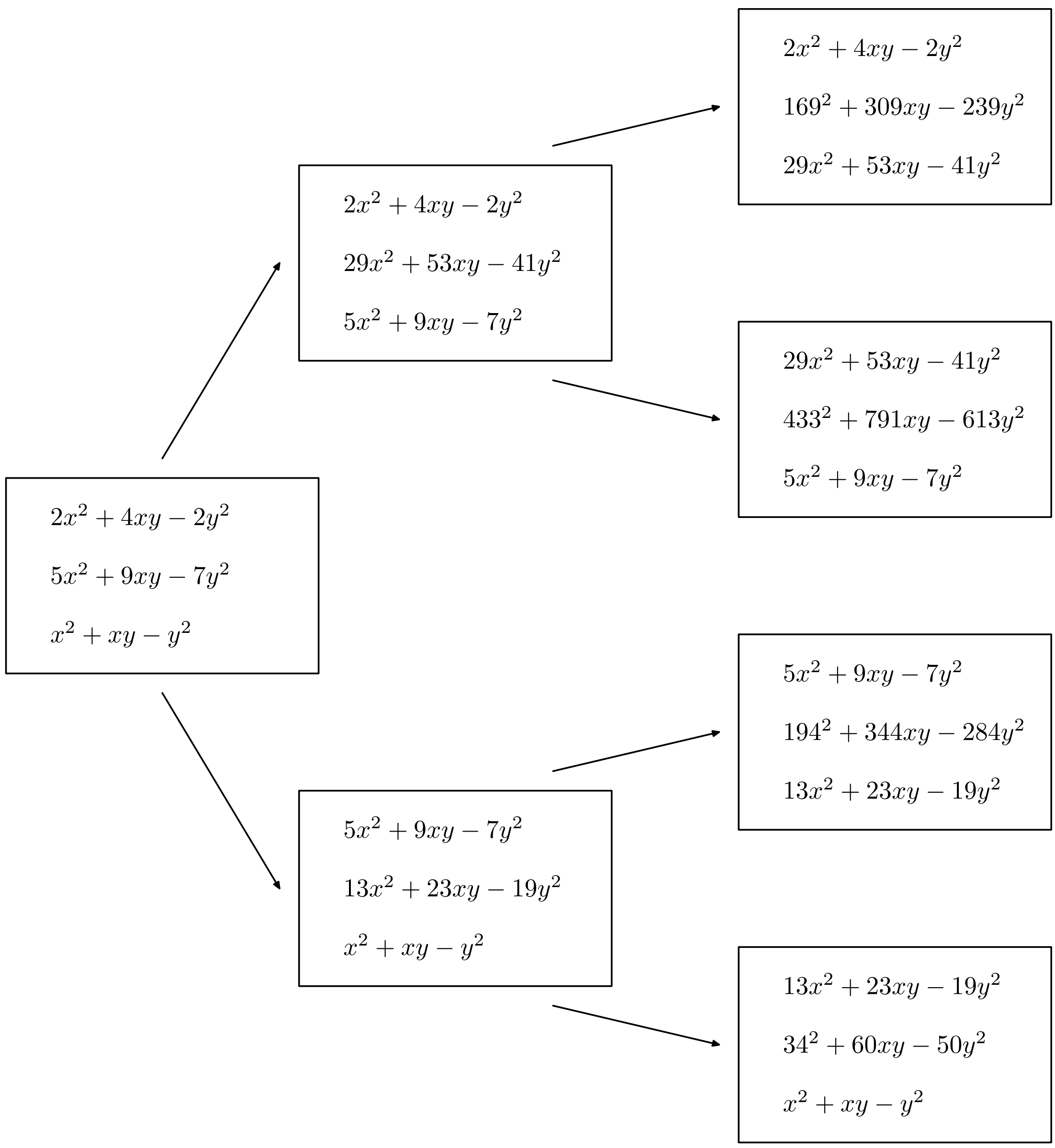

Figure 3.5: The forms from first 3 levels in the graph of regular reduced Markov forms.

where M_α and M_β are the matrices associated with the forms f_α and f_β, and α and β are even length sequences of positive integers.

Let

$$f_\alpha = A_1x^2 + A_2xy + A_3y^2, \qquad f_\beta = B_1x^2 + B_2xy + B_3y^2,$$
$$M_\alpha = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad M_\beta = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Then

$$\begin{aligned} A_1 &= a_{21}, & B_1 &= b_{21}, \\ A_2 &= a_{22} - a_{11}, & B_2 &= b_{22} - b_{11}, \\ A_3 &= -a_{12}, & B_3 &= -b_{12}. \end{aligned}$$

We have from Proposition 3.2.20 that $\widehat{\theta}(f_\alpha, f_\beta)$ is the form for the matrix

$$\begin{aligned} &\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} - A_3B_1 & -a_{11}B_3 - A_3B_2 - A_3b_{11} \\ A_1b_{11} + B_1A_2 + B_1a_{11} & -A_1B_3 + a_{11}b_{11} + A_2B_2 + a_{11}B_2 + b_{11}A_2 \end{pmatrix}. \end{aligned}$$

The form associated with this matrix is

$$\begin{aligned} \widehat{\theta}(f_\alpha, f_\beta) =& (A_1b_{11} + B_1A_2 + B_1a_{11})x^2 \\ &+ (-A_1B_3 + A_2B_2 + a_{11}B_2 + b_{11}A_2 + A_3B_1)xy \\ &+ (a_{11}B_3 + A_3B_2 + A_3b_{11})y^2. \end{aligned}$$

We define numbers A and B by the equations

$$\begin{aligned} a_{11} &= \frac{-A_2 + \sqrt{A_2^2 - 4(A_1A_3 - 1)}}{2} = \frac{-A_2 + A}{2}, \text{ and} \\ b_{11} &= \frac{-B_2 + \sqrt{B_2^2 - 4(B_1B_3 - 1)}}{2} = \frac{-B_2 + B}{2}. \end{aligned}$$

Then

$$\widehat{\theta}(f_\alpha, f_\beta) = C_1x^2 + C_2xy + C_3y^2,$$

where

$$\begin{aligned} C_1 &= \frac{-A_1B_2 + B_1A_2 + A_1B + B_1A}{2}, \\ C_2 &= \frac{AB_2 + BA_2}{2} + A_3B_1 - A_1B_3, \\ C_3 &= \frac{-A_2B_3 + AB_3 + A_3B_2 + A_3B}{2}. \end{aligned}$$

as required. This completes the proof.

■

Remark 3.2.32. From the proof of this theorem it follows for any even length sequences of positive integers μ and ν that

$$\widehat{\theta}\big(f_\mu, f_\nu\big) = f_{\mu\nu},$$

since the proof makes no other requirements on the sequences.

Table 3.4: The first 10 Markov numbers with their Markov forms and regular reduced Markov forms.

Markov triples (a,Q,b)	Markov form $f_{Q_{a,b}}$	Regular reduced form f_α
$(1,1,1)$	x^2+xy-y^2	x^2+xy-y^2
$(1,2,1)$	$2x^2+4xy-2y^2$	$2x^2+4xy-2y^2$
$(1,5,2)$	$5x^2+11xy-5y^2$	$5x^2+9xy-7y^2$
$(1,13,5)$	$13x^2+29xy-13y^2$	$13x^2+23xy-19y^2$
$(5,29,2)$	$29x^2+63xy-31y^2$	$29x^2+53xy-41y^2$
$(1,34,13)$	$34x^2+76xy-34y^2$	$34x^2+60xy-50y^2$
$(1,89,34)$	$89x^2+199xy-89y^2$	$89x^2+157xy-131y^2$
$(29,169,2)$	$169x^2+367xy-181y^2$	$169x^2+309xy-239y^2$
$(13,194,5)$	$194x^2+432xy-196y^2$	$194x^2+344xy-284y^2$
$(1,233,89)$	$233x^2+521xy-233y^2$	$233x^2+411xy-343y^2$

3.2.4 Matrix similarity and form equivalence

Recall that two forms f and g are *equivalent* if there exist integers a, b, c, and d with $ad-bc=\pm 1$ such that

$$f(x,y)=g(ax+by,cx+dy).$$

For a general reference on equivalence of integral forms see [18].

Recall also that two matrices A and B are *similar* if there exists an integer matrix P with $\det(P)=\pm 1$ such that

$$A=P^{-1}BP.$$

In this subsection we show that Markov forms and regular reduced Markov forms are equivalent. We also show that Cohn matrices and regular reduced Markov matrices are similar.

Markov form equivalence

Recall that $f_{Q_{a,b}}$ denotes the Markov form for the Markov triple (a,Q,b), as in Markov's theorem, Theorem 1.1.9.

Example 43. In Table 3.4 we compare Markov forms and regular reduced Markov forms for triples of the first ten Markov numbers. We see that the values at $(1,0)$ for both Markov forms and regular reduced Markov forms are equal for Markov numbers up to 233.

Denote the length of a sequence α by $|\alpha|$.

Proposition 3.2.33. *Let α be a regular Markov sequence, so that $Q=\breve{K}(\alpha)$ is a regular Markov number, and (a,Q,b) is some Markov triple. If $|\alpha|=2$ then*

$f_\alpha(x,y) = f_{Q_{a,b}}(x,y)$. If $|\alpha| > 2$ then we have the following equalities

$$\begin{aligned} f_\alpha(-x-2y,y) &= f_{Q_{a,b}}(x,y), \\ f_\alpha(x,y) &= f_{Q_{a,b}}(-x-2y,y). \end{aligned}$$

In particular, f_α and $f_{Q_{a,b}}$ are equivalent.

Proof. Note that for the sequences $(1,1)$ and $(2,2)$ we have

$$\begin{aligned} f_{(1,1)} &= x^2 + xy - y^2 = f_{1_{1,1}}, \qquad \text{and} \\ f_{(2,2)} &= 2x^2 + 4xy - 2y^2 = f_{2_{1,1}}. \end{aligned}$$

For $|\alpha| > 2$, equivalence of the forms $f_{Q_{a,b}}$ and f_α follows from proving either equality. The second equality follows from the first since the transformation matrix

$$\begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}$$

is self inverse. We prove the first equality.

We can write the coefficients of $f_{Q_{a,b}}$ (see for example [23, Theorem 3]) in terms of continuants of $\alpha = (a_1, \ldots, a_n)$ where $\breve{K}(\alpha) = Q$ and $n > 2$ as

$$\begin{aligned} 3Q - 2u &= 3K_1^{n-1}(\alpha) - 2K_1^{n-2}(\alpha), \\ v - 3u &= K_3^{n-2}(\alpha) - 3K_1^{n-2}(\alpha), \end{aligned}$$

where u and v are as in the definition of Markov forms (from Theorem 1.1.9). Comparing the coefficients of $f_\alpha(-x-2y,y)$ to those of $f_{Q_{a,b}}$ we see that the statement in this proposition is true if and only if the following equations hold

$$4K_1^{n-1}(\alpha) - K_1^n(\alpha) + K_2^{n-1}(\alpha) = 3K_1^{n-1}(\alpha) - 2K_1^{n-2}(\alpha), \tag{3.3}$$
$$4K_1^{n-1}(\alpha) - 2K_1^n(\alpha) + 2K_2^{n-1}(\alpha) - K_2^n(\alpha) = K_3^{n-2}(\alpha) - 3K_1^{n-2}(\alpha). \tag{3.4}$$

The left hand side of these equations are the xy and y^2 coefficients respectively of $f_\alpha(-x-2y,y)$, and the right hand side of these equations are the xy and y^2 coefficients respectively of $f_{Q_{a,b}}(x,y)$.

Since $\alpha = (1,1,a_3,\ldots,a_{n-2},2,2)$ we have by the recursive definition of continuants that $K_1^n(\alpha) = 2K_1^{n-1}(\alpha) + K_1^{n-2}(\alpha)$. Applying this to Equation (3.3) we get

$$\begin{aligned} K_1^{n-1}(\alpha) + K_2^{n-1}(\alpha) - K_1^n(\alpha) &= -2K_1^{n-2}(\alpha) \\ -K_1^{n-1}(\alpha) + K_2^{n-1}(\alpha) &= -K_1^{n-2}(\alpha) \\ K_1^{n-2}(\alpha) + K_2^{n-1}(\alpha) &= K_1^{n-1}(\alpha). \end{aligned}$$

Again by definition of continuants we have that $K_1^{n-1}(\alpha) = K_2^{n-1}(\alpha) + K_3^{n-1}(\alpha)$, and so

$$\begin{aligned} K_1^{n-2}(\alpha) + K_2^{n-1}(\alpha) &= K_1^{n-1}(\alpha) \\ K_1^{n-2}(\alpha) + K_2^{n-1}(\alpha) &= K_2^{n-1}(\alpha) + K_3^{n-1}(\alpha) \\ K_1^{n-2}(\alpha) &= K_3^{n-1}(\alpha). \end{aligned}$$

By Proposition 2.1.8 (iv) we have that

$$K_3^{n-1}(\alpha) = K_{n-3}(a_3, \dots, a_{n-2}, 2) = K_{n-2}(a_3, \dots, a_{n-2}, 1, 1).$$

From Proposition 3.2.5 we have that $(a_3, \dots, a_{n-2}) = (a_{n-2}, \dots, a_3)$. Combining this with the symmetric property of continuants (from Proposition 2.1.8 (ii)) we have

$$\begin{aligned} K_3^{n-1}(\alpha) &= K_{n-2}(a_3, \dots, a_{n-2}, 1, 1) \\ &= K_{n-2}(a_{n-2}, \dots, a_3, 1, 1) \\ &= K_{n-2}(1, 1, a_3, \dots, a_{n-2}) \\ &= K_1^{n-2}(\alpha). \end{aligned}$$

Applying similar continuant operations to Equation (3.4) we have

$$\begin{aligned} 4K_1^{n-1}(\alpha) - 2K_1^{n}(\alpha) + 2K_2^{n-1}(\alpha) &= K_3^{n-2}(\alpha) - 3K_1^{n-2}(\alpha) \\ -2K_1^{n-2}(\alpha) + 2K_2^{n-1}(\alpha) - K_2^{n}(\alpha) &= K_3^{n-2}(\alpha) - 3K_1^{n-2}(\alpha) \\ 2K_2^{n-1}(\alpha) - 2K_2^{n-1}(\alpha) - K_2^{n-2}(\alpha) &= K_3^{n-2}(\alpha) - K_1^{n-2}(\alpha) \\ -K_2^{n-2}(\alpha) &= K_3^{n-2}(\alpha) - K_1^{n-2}(\alpha), \end{aligned}$$

which is true by definition of continuants. This completes the proof. ■

Remark 3.2.34. Note that for a regular Markov sequence α we have that

$$m(f_\alpha) = f_\alpha(1, 0).$$

This follows since the forms f_α and $f_{Q_{a,b}}$ are equivalent for the Markov number $Q = \breve{K}(\alpha)$ in a Markov triple (a, Q, b), and

$$f_\alpha(1, 0) = f_{Q_{a,b}}(1, 0) = m\big(f_{Q_{a,b}}\big).$$

We finish this subsection with the following definition and corollary, relating the graphs of regular reduced Markov forms and of regular Markov numbers.

Definition 3.2.35. Define a map λ between triples forms and triples of integers by

$$\lambda(f_\alpha, f_\delta, f_\beta) = \big(f_\alpha(1, 0), f_\delta(1, 0), f_\beta(1, 0)\big).$$

Define a map Λ taking $\mathcal{G}_\theta\big(f_\mu, f_\nu\big)$ to a triple graph of integers in the following way: let vertices defined by triples v in $\mathcal{G}_\theta\big(f_\mu, f_\nu\big)$ be mapped to vertices with triples $\lambda(v)$, and let edges defined by the triples (v, w) be mapped to edges defined by the triples $\big(\lambda(v), \lambda(w)\big)$. Denote the new graph by $\Lambda\big(\mathcal{G}_\theta\big(f_\mu, f_\nu\big)\big)$.

Corollary 3.2.36. We have the following equality of graphs

$$\Lambda\big(\mathcal{G}_\theta\big(f_{(1,1)}, f_{(2,2)}\big)\big) = \mathcal{G}_\Sigma(1, 5, 2).$$

Proof. This follows from Proposition 3.2.33 since the values of a Markov form and its equivalent regular reduced Markov form are equal at $(1, 0)$, at which point the Markov forms attain their Markov minimum. ■

Markov matrix similarity

In this section we show that regular reduced Markov matrices are similar to Cohn matrices. The value in this comes from the following proposition, which is proven by O. Karpenkov in [52].

Proposition 3.2.37 (O. Karpenkov, [52])**.** *Two real spectrum* $\mathrm{SL}(2,\mathbb{Z})$ *matrices* A *and* B *with positive eigenvalues are* $\mathrm{SL}(2,\mathbb{Z})$ *similar if and only if their LLS periods coincide.*

Example 44. We note that the regular reduced matrices for the Markov sequences $(1,1)$ and $(2,2)$

$$M_{(1,1)} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad M_{(2,2)} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

are both Cohn matrices, but do not constitute a starting pair of Cohn matrices. They are symmetric, so the Markov number appears in both the upper right and lower left positions.

Definition 3.2.38. Let P be the matrix defined by

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that $\det(P) = -1$. Define a map π_P from triples of matrices to triples of matrices by

$$\pi_P\big(M_\alpha, M_\delta, M_\beta\big) = \big(P^{-1}M_\alpha P, P^{-1}M_\delta P, P^{-1}M_\beta P\big).$$

Define a map Π_P from a graph $\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)$ to a triple graph of $\mathrm{SL}(2,\mathbb{Z})$ matrices in the following way: let vertices defined by triples v in $\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)$ be mapped to vertices with triples $\pi_P(v)$, and let edges defined by the triples (v,w) be mapped to edges defined by the triples $\big(\pi_P(v), \pi_P(w)\big)$. Denote this new graph by $\Pi_P\big(\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)\big)$.

Recall the definition of $\mathcal{G}_C(a)$, the graph of Cohn matrices defined by the integer a.

Proposition 3.2.39. *We have that*

$$\Pi_P\big(\mathcal{G}_\bullet\big(M_{(1,1)}, M_{(2,2)}\big)\big) = \mathcal{G}_C(1).$$

Proof. We use induction. By inspection we have that triples of matrices

$$\big(M_{(1,1)}, M_{(1,1,2,2)}, M_{(2,2)}\big) \quad \text{and} \quad \big(C_1(1), C_1(1)C_2(1), C_2(1)\big)$$

are similar, with transformation matrix

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

This provides a base of induction. Recall for matrices A, B, M, and N and X invertible that if

$$X^{-1}AX = M \quad \text{and} \quad X^{-1}BX = N$$

then

$$X^{-1}ABX = X^{-1}AXX^{-1}BX = MN.$$

Assume for some vertices $\big(M_\alpha, M_{\alpha\beta}, M_\beta\big)$ in $\mathcal{G}_\bullet\big(M_{(1,1)}, M_{(2,2)}\big)$ and $\big(C_a, C_aC_b, C_b\big)$ in $\mathcal{G}_C(1)$, that

$$\big(P^{-1}M_\alpha P, P^{-1}M_{\alpha\beta}P, P^{-1}M_\beta P\big) = \big(C_a, C_aC_b, C_b\big).$$

Then

$$\begin{aligned}\big(P^{-1}M_\alpha P, P^{-1}M_{\alpha\alpha\beta}P, P^{-1}M_{\alpha\beta}P\big) &= \big(C_a, C_a^2C_b, C_aC_b\big),\\ \big(P^{-1}M_{\alpha\beta} P, P^{-1}M_{\alpha\beta\beta}P, P^{-1}M_{\beta}P\big) &= \big(C_aC_b, C_aC_b^2, C_b\big).\end{aligned}$$

This proves the induction, and the proposition follows. ∎

We end this subsection with the following definition and corollary, relating the graphs of regular Markov numbers and regular reduced Markov matrices.

Definition 3.2.40. For a matrix

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix},$$

denote the element $a_{i,j}$ by $M(i,j)$.

(i) Define a map υ from triples of matrices to triples of integers by

$$\upsilon\big(M_\alpha, M_{\alpha\beta}, M_\beta\big) = \big(M_\alpha(2,1), M_{\alpha\beta}(2,1), M_\beta(2,1)\big).$$

Define a map Υ from a graph $\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)$ to a triple graph of integers in the following way: let vertices defined by triples w_1 in $\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)$ be mapped to vertices with triples $\upsilon(w_1)$, and let edges defined by the triples (w_1, w_2) be mapped to edges defined by the triples $\big(\upsilon(w_1), \upsilon(w_2)\big)$. Denote this new graph by $\Upsilon\big(\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)\big)$.

(ii) Define a map υ_T from triples of matrices to triples of integers by

$$\upsilon_T\big(M_\alpha, M_{\alpha\beta}, M_\beta\big) = \left(\frac{\mathrm{tr}(M_\alpha)}{3}, \frac{\mathrm{tr}(M_{\alpha\beta})}{3}, \frac{\mathrm{tr}(M_\beta)}{3}\right).$$

Define a map Υ_T from a graph $\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)$ to a triple graph of integers in the following way: let vertices defined by triples v in $\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)$ be mapped to vertices with triples $\upsilon_T(v)$, and let edges defined by the triples (v, w) be mapped to edges defined by the triples $\big(\upsilon_T(v), \upsilon_T(w)\big)$. Denote this new graph by $\Upsilon_T\big(\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)\big)$.

Corollary 3.2.41. We have the following equalities of graphs.

$$\Upsilon_T\big(\mathcal{G}_\bullet\big(M_{(1,1)}, M_{(2,2)}\big)\big) = G_\Sigma(1,5,2),$$
$$\Upsilon\big(\mathcal{G}_\bullet\big(M_{(1,1)}, M_{(2,2)}\big)\big) = G_\Sigma(1,5,2).$$

Proof. Matrix similarity preserves the trace. The result then follows from Proposition 3.2.39. ∎

We finish this subsection by noting that Proposition 3.2.7 follows as a corollary to Proposition 3.2.39.

Proof of Proposition 3.2.7. Proposition 3.2.7 follows from the similarity of Cohn matrices and regular reduced Markov matrices. ∎

3.3 Graph relations

In this section we discuss relations between the previously defined triple graphs of regular Markov numbers, matrices, forms, and sequences.

In Subsection 3.3.1 we define a map Ω between triple graphs of matrices and forms. We also define its inverse map Ω^{-1}.

In Subsection 3.3.2 we define the inverse maps to Φ and Γ from Section 3.2.

3.3.1 Relation between forms and matrices

In this subsection we define maps between triple graphs of matrices and forms. We base these maps on the result of Proposition 3.2.11. Recall that for a real valued matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with determinant $|M|$ and $\operatorname{tr}(M)^2 > 4|M|$, the form associated with M is

$$f_M(x, y) = cx^2 + (d - a)xy - by^2.$$

We define a map from matrices to forms.

Definition 3.3.1. Let ω be the map between triples of matrices and triples of forms in $\mathcal{F}^3$ defined by

$$\omega(M, MN, N) = \big(f_M, f_{MN}, f_N\big).$$

Define a map Ω from a graph of matrices $\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)$ to a graph of forms in the following way: let vertices defined by triples v in $\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)$ be mapped to vertices with triples $\omega(v)$, and let edges defined by the triples (v, w) be mapped to edges defined by the triples $\big(\omega(v), \omega(w)\big)$. Denote the new graph by

$$\Omega\big(\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)\big).$$

Remark 3.3.2. Note that by the definition of the graph $\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)$, any matrix M in any vertex has $\det(M) = 1$ and a trace greater than one. Hence they satisfy the condition that $\operatorname{tr}(M)^2 > |M|$ from Proposition 3.2.11.

Remark 3.3.3. Recall the standard definition way to associate a matrix with a form $f(x, y) = a_1x^2 + a_2xy + a_3y^2$,

$$\begin{pmatrix} a_1 & \frac{a_2}{2} \\ \frac{a_2}{2} & a_3 \end{pmatrix}.$$

In this thesis we use a different definition for a matrix associated with a form.

Definition 3.3.4. For a form $f(x,y) = a_1x^2 + a_2xy + a_3y^2$ define the *matrix associated with* f to be

$$M_f = \begin{pmatrix} \frac{-a_2+\sqrt{a_2^2-4(a_1a_3-1)}}{2} & -a_3 \\ a_1 & \frac{a_2+\sqrt{a_2^2-4(a_1a_3-1)}}{2} \end{pmatrix}.$$

We use this notion to define an inverse map to Ω.

Definition 3.3.5. Define a map ω^{-1} taking triples of forms in $\mathcal{F}$ to triples of $\mathrm{SL}(2,\mathbb{Z})$ matrices by

$$\omega^{-1}(f,h,g) = \big(M_f, M_h, M_g\big).$$

Define a map Ω^{-1} from a graph of forms $\mathcal{G}_\theta\big(f_\mu, f_\nu\big)$ to a graph of matrices in the following way: let vertices defined by triples v in $\mathcal{G}_\theta\big(f_\mu, f_\nu\big)$ be mapped to vertices with triples $\omega^{-1}(v)$, and let edges defined by the triples (v,w) be mapped to edges defined by the triples $\big(\omega^{-1}(v), \omega^{-1}(w)\big)$. Denote the new graph by $\Omega^{-1}\big(\mathcal{G}_\theta\big(f_\mu, f_\nu\big)\big)$.

Remark 3.3.6. Note for an even sequence $\alpha = (a_1, \ldots, a_n)$ that $|M_\alpha| = 1$ and

$$\mathrm{tr}\,\big(M_\alpha\big) = K_2^{n-1}(\alpha) + K_1^n(\alpha) > 2.$$

Hence $\mathrm{tr}\,\big(M_\alpha\big)^2 > 4|M_\alpha|$.

We show that Ω and Ω^{-1} are maps between the graphs of regular reduced Markov forms and regular reduced Markov matrices. We show also that they are inverse maps.

Proposition 3.3.7. *For even sequences μ and ν such that the matrices M_μ, $M_{\mu\nu}$, and M_ν all satisfy the condition $\mathrm{tr}(M)^2 > |M|$, and hence $\mathrm{tr}(M)^2 > 4|M|$, we have the following equality of graphs*

$$\Omega\big(\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)\big) = \mathcal{G}_\theta\big(f_\mu, f_\nu\big),$$
$$\mathcal{G}_\bullet\big(M_\mu, M_\nu\big) = \Omega^{-1}\big(\mathcal{G}_\theta\big(f_\mu, f_\nu\big)\big).$$

Moreover, the map Ω is a graph isomorphism with inverse Ω^{-1}.

Remark 3.3.8. In particular, Proposition 3.3.7 says that

$$\Omega\big(\mathcal{G}_\bullet\big(M_{(1,1)}, M_{(2,2)}\big)\big) = \mathcal{G}_\theta(f_{(1,1)}, f_{(2,2)})$$

and

$$\mathcal{G}_\bullet\big(M_{(1,1)}, M_{(2,2)}\big) = \Omega^{-1}\big(\mathcal{G}_\theta(f_{(1,1)}, f_{(2,2)})\big).$$

Lemma 3.3.9. Let α and β be sequences of even length such that the matrices M_α, $M_{\alpha\beta}$, and M_β satisfy the condition $\mathrm{tr}(M)^2 > |M|$. Then

$$f_{\alpha\beta} = f_{M_{\alpha\beta}} = \widehat{\theta}\big(f_\alpha, f_\beta\big).$$

Proof. By definition we have that

$$f_{\alpha\beta} = [K_1^{n+m-1}(\alpha\beta), K_1^{n+m}(\alpha\beta) - K_2^{n+m-1}(\alpha\beta), K_2^{n+m}(\alpha\beta)].$$

Define real numbers A and B by

$$\begin{aligned} A &= +\sqrt{\big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)^2 - 4\big(\breve{K}(\alpha)K_2^n(\alpha) - 1\big)}, \\ B &= +\sqrt{\big(K_1^m(\beta) - K_2^{m-1}(\beta)\big)^2 - 4\big(\breve{K}(\beta)K_2^m(\beta) - 1\big)}. \end{aligned}$$

Note that

$$\begin{aligned} A &= \sqrt{\big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)^2 - 4\big(\breve{K}(\alpha)K_2^n(\alpha) - 1\big)} \\ &= \sqrt{\big(K_1^n(\alpha) + K_2^{n-1}(\alpha)\big)^2 - 4K_1^n(\alpha)K_2^{n-1}(\alpha) - 4\breve{K}(\alpha)K_2^n(\alpha) + 4} \\ &= \sqrt{\big(K_1^n(\alpha) + K_2^{n-1}(\alpha)\big)^2} \\ &= K_1^n(\alpha) + K_2^{n-1}(\alpha). \end{aligned}$$

Similarly, $B = K_1^m(\beta) + K_2^{m-1}(\beta)$. We have that $\widehat{\theta}\big(f_\alpha, f_\beta\big) = c_1x^2 + c_2xy + c_3y^2$ from Definition 3.2.30, where

$$\begin{aligned} c_1 &= \frac{\breve{K}(\alpha)\big(B - \big(K_1^m(\beta) - K_2^{m-1}(\beta)\big)\big) + \breve{K}(\beta)\big(A + \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)\big)}{2}, \\ c_2 &= \frac{B\big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big) + A\big(K_1^m(\beta) - K_2^{m-1}(\beta)\big)}{2} - \breve{K}(\alpha)K_2^m(\beta) + K_2^n(\alpha)\breve{K}(\beta), \\ c_3 &= \frac{K_2^m(\beta)\big(A - \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)\big) + K_2^n(\alpha)\big(B + \big(K_1^m(\beta) - K_2^{m-1}(\beta)\big)\big)}{2}. \end{aligned}$$

We have that

$$\begin{aligned} c_1 &= K_1^{n-1}(\alpha)K_2^{m-1}(\beta) + K_1^n(\alpha)K_1^{m-1}(\beta) \\ &= K_1^{n+m-1}(\alpha\beta). \end{aligned}$$

For c_3 we have that

$$\begin{aligned} c_3 &= K_2^{n-1}(\alpha)K_2^m(\beta) + K_2^n(\alpha)K_1^m(\beta) \\ &= K_2^{n+m}(\alpha\beta). \end{aligned}$$

Finally, for c_2 we have

$$\begin{aligned} c_2 &= K_1^n(\alpha)K_1^m(\beta) - K_2^{n-1}(\alpha)K_2^{m-1}(\beta) - K_1^{n-1}(\alpha)K_2^m(\beta) + K_2^n(\alpha)K_1^{m-1}(\beta) \\ &= K_1^{m+n-1}(\beta\alpha) - K_2^{m+n-1}(\beta\alpha). \end{aligned}$$

From the symmetric property of continuants, we have that

$$K_1^{m+n-1}(\beta\alpha) - K_2^{m+n-1}(\beta\alpha) = K_1^{n+m-1}(\alpha\beta) - K_2^{n+m-1}(\alpha\beta).$$

This concludes the proof. ∎

Now we complete the proof of Proposition 3.3.7

$$
\begin{tikzcd}
\boxed{\text{Regular Markov sequences}} \arrow[rr, "\Gamma"] \arrow[dr, "\Phi"'] & & \boxed{\text{Regular reduced Markov matrices}} \arrow[dl, shift left, "\Omega"] \\
& \boxed{\text{Regular reduced Markov forms}} \arrow[ur, shift left, "\Omega^{-1}"] &
\end{tikzcd}
$$

Figure 3.6: Relations between regular sequences and matrices, and regular sequences and forms.

Proof of Proposition 3.3.7. By definition we have that

$$\omega\big(M_\mu, M_{\mu\nu}, M_\nu\big) = \big(f_\mu, f_{\mu\nu}, f_\nu\big).$$

This is a base of induction. We assume that $\omega\big(M_\alpha, M_{\alpha\beta}, M_\beta\big) = \big(f_\alpha, f_{\alpha\beta}, f_\beta\big)$ for some vertex $\big(M_\alpha, M_{\alpha\beta}, M_\beta\big)$ in $\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)$. We show that

$$\begin{aligned}
\omega\big(M_\alpha, M_{\alpha\alpha\beta}, M_{\alpha\beta}\big) &= \big(f_\alpha, f_{\alpha\alpha\beta}, f_{\alpha\beta}\big),\\
\omega\big(M_{\alpha\beta}, M_{\alpha\beta\beta}, M_\beta\big) &= \big(f_{\alpha\beta}, f_{\alpha\beta\beta}, f_\beta\big).
\end{aligned}$$

Both of these equalities follow directly from Lemma 3.3.9.

Similar to Ω, to show the inverse map Ω^{-1} we only need to show the equality

$$\omega^{-1}\big(f_\alpha, f_{\alpha\beta}, f_\beta\big) = \big(M_\alpha, M_{\alpha\beta}, M_\beta\big).$$

We have that $\omega^{-1}\big(f_\alpha, f_{\alpha\beta}, f_\beta\big) = (M_1, M_2, M_3)$, where

$$M_2 = \begin{pmatrix} K_2^{n+m-1}(\alpha\beta) & K_2^{n+m}(\alpha\beta) \\ K_1^{n+m-1}(\alpha\beta) & K_1^{n+m}(\alpha\beta) \end{pmatrix}.$$

The upper right and bottom left elements of the matrix are derived in the proof of Lemma 3.3.9. This matrix is exactly $M_{\alpha\beta}$, concluding the proof. ∎

3.3.2 Inverse maps

Recall the maps Φ and Γ above. We have the relations between regular Markov sequences, regular reduced Markov matrices, and regular reduced Markov matrices in Figure 3.6. We define maps Φ^{-1} and Γ^{-1}. We show in Proposition 3.3.14 that Φ and Φ^{-1} are inverse, as are Γ and Γ^{-1}.

Definition 3.3.10. For an $\mathrm{SL}(2,\mathbb{Z})$ matrix
$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
define a map $\widehat{\varepsilon}$ by
$$\widehat{\varepsilon}(M) = \left[\frac{d}{b}\right],$$
where $[d/b]$ denotes the sequence in the even regular continued fraction expansion of d/b.

Define a map ε from triples of matrices to triples of finite sequences by
$$\varepsilon(M_\alpha, M_{\alpha\beta}, M_\beta) = \big(\widehat{\varepsilon}(M_\alpha), \widehat{\varepsilon}(M_{\alpha\beta}), \widehat{\varepsilon}(M_\beta)\big).$$

Define a map Γ^{-1} between $\mathcal{G}_\bullet(M_\mu, M_\nu)$ and a graph of triples of sequences in the following way: let vertices defined by triples v in $\mathcal{G}_\bullet(M_\mu, M_\nu)$ be mapped to vertices with triples $\varepsilon(v)$, and let edges defined by the triples (v, w) be mapped to edges defined by the triples $\big(\varepsilon(v), \varepsilon(w)\big)$. Denote this graph by $\Gamma^{-1}\big(\mathcal{G}_\bullet(M_\mu, M_\nu)\big)$.

Recall the following classical remark from Subsection 2.2.2, proven in the book by O. Karpenkov [52].

Remark 3.3.11. Let $f(x,y) = (y - ax)(y - bx)$ be a reduced form with $a > 1$ and $-1 < b < 0$. Let the continued fraction expansions of a and b be
$$a = [a_1; a_2 : \ldots] = [\alpha],$$
$$b = -[0; b_1 : b_2 : \ldots] = [0; \beta],$$
for some infinite sequences of positive integers α and β. Then the LLS sequence of f is
$$\overline{\beta}\alpha.$$

From the classical theory of forms we have the following corollary, which allows us to extract the sequence α from the roots of the quadratic $f_\alpha(x,1)$.

Corollary 3.3.12. For each form in a triple $(f_\alpha, f_{\alpha\beta}, f_\beta)$ we have that
$$f_\alpha(x,1) = \breve{K}(\alpha)\big(x - \big[0;(\alpha)\big]\big)\big(x + \big[(\overline{\alpha})\big]\big),$$
$$f_{\alpha\beta}(x,1) = \breve{K}(\alpha\beta)\big(x - \big[0;(\alpha\beta)\big]\big)\big(x + \big[(\overline{\alpha\beta})\big]\big),$$
$$f_\beta(x,1) = \breve{K}(\beta)\big(x - \big[0;(\beta)\big]\big)\big(x + \big[(\overline{\beta})\big]\big).$$

Definition 3.3.13. We define a map φ^{-1} from such triples of forms to triples of sequences by
$$\varphi^{-1}(f_\alpha, f_{\alpha\beta}, f_\beta) = (\alpha, \alpha\beta, \beta).$$

Define a map Φ^{-1} between $\mathcal{G}_\theta(f_\mu, f_\nu)$ and a graph of triples of sequences in the following way: let vertices defined by triples v in $\mathcal{G}_\theta(f_\mu, f_\nu)$ be mapped to vertices with triples $\varphi^{-1}(v)$, and let edges defined by the triples (v, w) be mapped to edges defined by the triples $\big(\varphi^{-1}(v), \varphi^{-1}(w)\big)$. Denote this graph by $\Phi^{-1}\big(\mathcal{G}_\theta(f_\mu, f_\nu)\big)$.

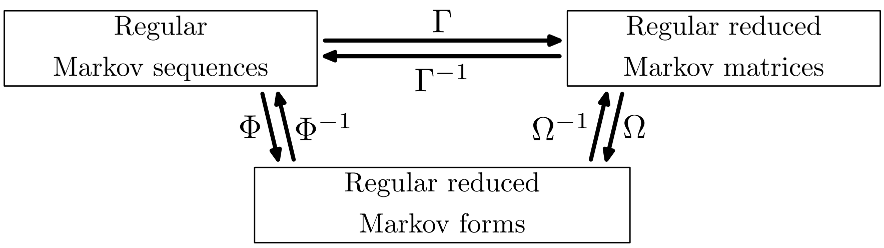

Figure 3.7: Relations between regular Markov sequences, matrices, and forms.

Proposition 3.3.14. *Let μ and ν be even length sequences of positive integers. Then*

$$\Gamma^{-1}\big(\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)\big) = \mathcal{G}_\oplus(\mu,\nu),$$
$$\Phi^{-1}\big(\mathcal{G}_\theta\big(f_\mu, f_\nu\big)\big) = \mathcal{G}_\oplus(\mu,\nu).$$

In particular, the maps Γ and Φ are graph isomorphisms, with inverses Γ^{-1} and Φ^{-1} respectively.

Corollary 3.3.15. In particular, we have in the case for regular Markov matrices, forms, and sequences that

$$\Gamma^{-1}\big(\mathcal{G}_\bullet\big(M_{(1,1)}, M_{(2,2)}\big)\big) = \mathcal{G}_\oplus\big((1,1),(2,2)\big),$$
$$\Phi^{-1}\big(\mathcal{G}_\theta\big(f_{(1,1)}, f_{(2,2)}\big)\big) = \mathcal{G}_\oplus\big((1,1),(2,2)\big).$$

From this corollary we have the diagram in Figure 3.7 showing the inverse maps Γ^{-1} and Φ^{-1}.

Lemma 3.3.16. Let μ and ν be even length sequences of positive integers. Let $v = \big(M_\alpha, M_{\alpha\beta}, M_\beta\big)$, with $\alpha = (a_1,\dots,a_n)$ and $\beta = (b_1,\dots,b_m)$, be the triple at some vertex in $\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)$. Then

$$\varepsilon(v) = (\alpha, \alpha\beta, \beta).$$

For any $\alpha = (a_1,\dots,a_n)$ in any vertex of $\mathcal{G}_\oplus(\mu,\nu)$ we have that

$$f_\alpha(x,y) = f_{M_\alpha}(x,y).$$

Proof. We have that

$$\widehat{\varepsilon}\big(M_\alpha\big) = \left[\frac{K_1^n(\alpha)}{K_2^n(\alpha)}\right] = \alpha,$$
$$\widehat{\varepsilon}\big(M_\beta\big) = \left[\frac{K_1^m(\beta)}{K_2^m(\beta)}\right] = \beta,$$
$$\widehat{\varepsilon}\big(M_{\alpha\beta}\big) = \left[\frac{K_1^{n+m}(\alpha\beta)}{K_2^{n+m}(\alpha\beta)}\right] = \alpha\beta.$$

Furthermore, we have by definition that

$$f_{M_\alpha}(x,y) = K_1^{n-1}(\alpha)x^2 + \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)xy - K_2^n(\alpha)y^2.$$

This completes the proof. ∎

We can now prove Proposition 3.3.14.

Proof of Proposition 3.3.14. Note that the equality

$$\Phi^{-1}\big(\mathcal{G}_\theta\big(f_\mu, f_\nu\big)\big) = \mathcal{G}_\oplus(\mu, \nu)$$

follows from the definition of general reduced Markov forms.

For the other equality we use induction. We have that

$$\varepsilon\big(M_\mu, M_{\mu\nu}, M_\nu\big) = (\mu, \mu\nu, \nu)$$

by inspection. This is the base of induction.

Assume $\varepsilon(v) = (\alpha, \alpha\beta, \beta)$ for some vertex $v = \big(M_\alpha, M_{\alpha\beta}, M_\beta\big)$ in $\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)$. We show that

$$\varepsilon\big(M_\alpha, M_{\alpha\alpha\beta}, M_{\alpha\beta}\big) = (\alpha, \alpha\alpha\beta, \alpha\beta),$$
$$\varepsilon\big(M_{\alpha\beta}, M_{\alpha\beta\beta}, M_\beta\big) = (\alpha\beta, \alpha\beta\beta, \beta).$$

Both equalities follow directly from Lemma 3.3.16. This proves the induction.

■

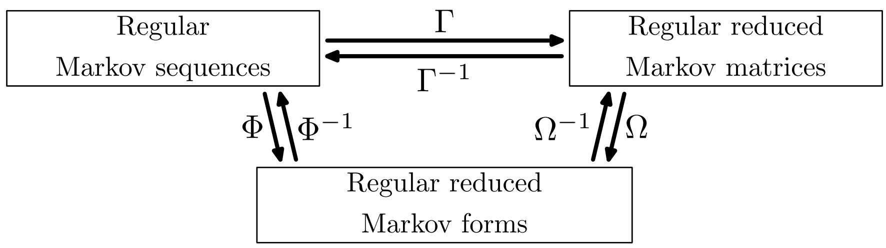

Figure 3.8: Relations between regular Markov sequences, matrices, and forms.

3.4 Diagram of Classical Markov theory

We expand upon the diagram in Figure 3.8 relating regular Markov sequences, regular reduced Markov matrices, and regular reduced Markov forms.

Recall the map X from Subsection 3.2.1, and the maps Λ and Υ from Subsection 3.2.4. These relate Markov sequences, matrices, and forms to regular Markov numbers. In Subsection 3.4.1 we define maps Ψ_i, for $i \in \{1, 2, 3, 4\}$, from graphs of regular Markov numbers, matrices, forms, and sequences to the Markov spectrum, completing the diagram in Figures 3.9 and 3.10.

In Subsection 3.4.2 we simplify the Perron identity for periodic sequences. By using a Markov sequence δ we remove the need to evaluate the supremum of the set of values

$$\sup\left\{[\langle\overline{\delta_i}\rangle] + [0; \langle\delta_i\rangle] \mid i \in \mathbb{Z}\right\},$$

where δ_i is the i-th cyclic shift of δ. The Perron identity is instead simply given by

$$\left[\langle\overline{\delta}\rangle\right] + [0; \langle\delta\rangle].$$

While this result is known at least since Frobenius [38], it is worth writing in detail since, in later chapters, the result is extended to any sequence contained

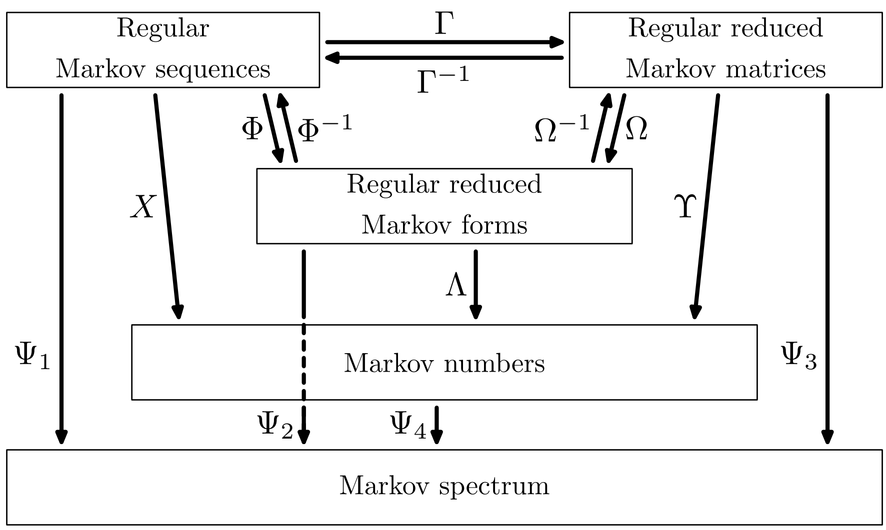

Figure 3.9: The classical structure of Markov theory.

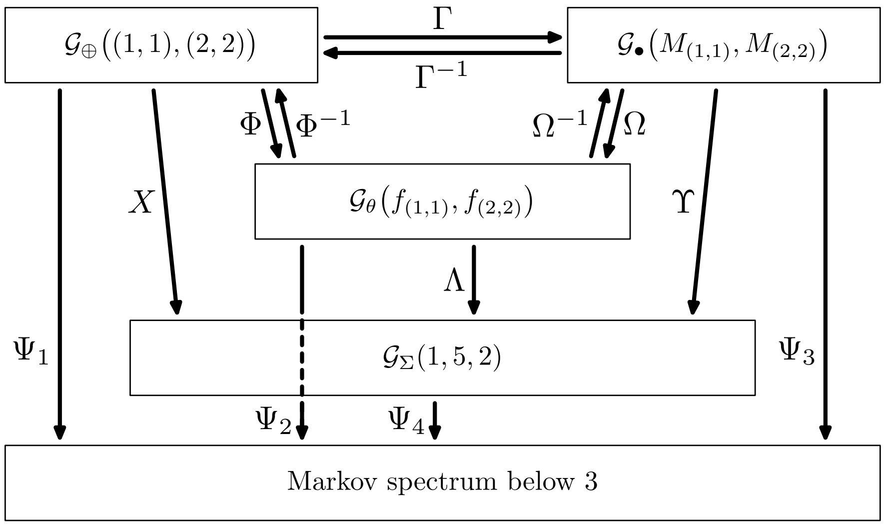

Figure 3.10: The classical structure of Markov theory.

in a Markov graph (see Subsection 4.4.2 for details).

In Subsection 3.4.3 we show that inverse maps to X, Λ, and Υ exist if and only if the uniqueness conjecture holds for regular Markov numbers.

3.4.1 Markov spectrum maps

In this subsection we define maps between graphs of regular Markov numbers, matrices, forms, and sequences with the Markov spectrum. We begin with a definition.

Definition 3.4.1.

(i) Define the map ρ_1 acting on vertices of a graph $\mathcal{G}_\oplus(\mu,\nu)$ by

$$\rho_1(\alpha,\delta,\beta) = \frac{\sqrt{\big(K_1^n(\delta)+K_2^{n-1}(\delta)\big)^2-4}}{K_1^{n-1}(\delta)},$$

where $\delta = (d_1,\ldots,d_n)$ and $n \geq 2$.

Define the map Ψ_1 taking $\mathcal{G}_\oplus(\mu,\nu)$ to a subset of $\mathbb{R}$ where the triples v at vertices are mapped to $\rho_1(v)$. Denote this subset of $\mathbb{R}$ by $\Psi_1\big(\mathcal{G}_\oplus(\mu,\nu)\big)$.

(ii) Define the map ρ_2 acting on vertices of a graph $\mathcal{G}_\omega\big(f_\mu, f_\nu\big)$ by

$$\rho_2(f_\alpha, f_\delta, f_\beta) = \frac{\sqrt{\Delta(f_\delta)}}{m(f_\delta)}.$$

Define the map Ψ_2 taking $\mathcal{G}_\theta(f_\mu, f_\nu)$ to a subset of $\mathbb{R}$ where the triples v at vertices are mapped to $\rho_2(v)$. Denote this subset of $\mathbb{R}$ by $\Psi_2\big(\mathcal{G}_\theta(f_\mu, f_\nu)\big)$.

(iii) Define the map ρ_3 acting on vertices of a graph $\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)$ by

$$\rho_3(M_\alpha, M_\delta, M_\beta) = \frac{\sqrt{\operatorname{tr}(M_\delta)^2 - 4}}{\upsilon(M_{\alpha\beta})}.$$

Define the map Ψ_3 taking $\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)$ to a subset of $\mathbb{R}$ where the triples v at vertices are mapped to $\rho_3(v)$. Denote this subset of $\mathbb{R}$ by $\Psi_3\big(\mathcal{G}_\bullet\big(M_\mu, M_\nu\big)\big)$.

(iv) Define the map ρ_4 acting on vertices of a graph $\mathcal{G}_\Sigma(x, y, z)$ by

$$\rho_4(a, Q, b) = \frac{\sqrt{9Q^2 - 4}}{Q}.$$

Define the map Ψ_4 taking $\mathcal{G}_\Sigma(x, y, z)$ to a subset of $\mathbb{R}$ where the triples v at vertices are mapped to $\rho_4(v)$.

Proposition 3.4.2. *Denote the Markov spectrum below 3 by $\mathcal{M}_{<3}$. Then*

$$\begin{aligned}
\Psi_4\big(\mathcal{G}_\Sigma(1,5,2)\big) &= \mathcal{M}_{<3},\\
\Psi_1\big(\mathcal{G}_\oplus\big((1,1),(2,2)\big)\big) &= \mathcal{M}_{<3},\\
\Psi_2\big(\mathcal{G}_\theta\big(f_{(1,1)}, f_{(2,2)}\big)\big) &= \mathcal{M}_{<3},\\
\Psi_3\big(\mathcal{G}_\bullet\big(M_{(1,1)}, M_{(2,2)}\big)\big) &= \mathcal{M}_{<3}.
\end{aligned}$$

Lemma 3.4.3. For a sequence $\alpha = (a_1, \ldots, a_n)$ in any vertex of $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$ with $n > 2$ we have that

$$\frac{K_1^{2n-1}(\alpha^2)}{K_1^{n-1}(\alpha)} = 3K_1^{n-1}(\alpha).$$

Proof. Since the subsequence $(a_3, \ldots, a_{n-2})$ is symmetric we have that

$$K_1^{n-2}(\alpha) = K(1, 1, a_3, \ldots, a_{n-2}) = K(2, a_3, \ldots, a_{n-2}) = K(a_3, \ldots, a_{n-2}, 2).$$

Hence

$$\begin{aligned}
\frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)} &= K(\alpha) + K_2^{n-1}(\alpha)\\
&= 2\breve{K}(\alpha) + K_1^{n-2}(\alpha) + K_2^{n-1}(\alpha)\\
&= 2\breve{K}(\alpha) + K_3^{n-1}(\alpha) + K_2^{n-1}(\alpha)\\
&= 3\breve{K}(\alpha).
\end{aligned}$$

This completes the proof. ■

Lemma 3.4.4. For $i \in \{1, 2, 3\}$ each Ψ_i may be written in the following way

$$\begin{aligned}
\Psi_1\big(\mathcal{G}_\oplus\big((1,1),(2,2)\big)\big) &= \Psi_4 \circ X\big(\mathcal{G}_\oplus\big((1,1),(2,2)\big)\big),\\
\Psi_2\big(\mathcal{G}_\theta\big(f_{(1,1)}, f_{(2,2)}\big)\big) &= \Psi_4 \circ \Lambda\big(\mathcal{G}_\theta\big(f_{(1,1)}, f_{(2,2)}\big)\big),\\
\Psi_3\big(\mathcal{G}_\bullet\big(M_{(1,1)}, M_{(2,2)}\big)\big) &= \Psi_4 \circ \Upsilon\big(\mathcal{G}_\bullet\big(M_{(1,1)}, M_{(2,2)}\big)\big).
\end{aligned}$$

Proof. We show the following equalities:

$$\begin{aligned}
\rho_1(\alpha,\delta,\beta) &= \rho_4\big(\breve{K}(\alpha),\breve{K}(\delta),\breve{K}(\beta)\big),\\
\rho_2(f_\alpha,f_\delta,f_\beta) &= \rho_4\big(f_\alpha(1,0),f_\delta(1,0),f_\beta(1,0)\big),\\
\rho_3(M_\alpha,M_\delta,M_\beta) &= \rho_4\big(\upsilon(M_\alpha),\upsilon(M_\delta),\upsilon(M_\beta)\big).
\end{aligned}$$

From the result in Lemma 3.4.3 we have for $\delta=(d_1,\ldots,d_n)$, a regular Markov sequence, that

$$K_1^n(\delta)+K_2^{n-1}(\delta)=3\breve{K}(\delta).$$

For ρ_1 we then have

$$\begin{aligned}
\rho_1(\alpha,\delta,\beta) &= \frac{\sqrt{\big(K_1^n(\delta)+K_2^{n-1}(\delta)\big)^2-4}}{K_1^{n-1}(\delta)}\\
&= \frac{\sqrt{9\breve{K}(\delta)^2-4}}{\breve{K}(\delta)}\\
&= \rho_4\big(\breve{K}(\alpha),\breve{K}(\delta),\breve{K}(\beta)\big).
\end{aligned}$$

For ρ_2 we have

$$\begin{aligned}
\rho_2(f_\alpha,f_\delta,f_\beta) &= \frac{\sqrt{\Delta(f_\delta)}}{m(f_\delta)}\\
&= \frac{\sqrt{\big(K_1^n(\delta)+K_2^{n-1}(\delta)\big)^2-4}}{f_\delta(1,0)}\\
&= \frac{\sqrt{9f_\delta(1,0)^2-4}}{f_\delta(1,0)}\\
&= \rho_4\big(f_\alpha(1,0),f_\delta(1,0),f_\beta(1,0)\big).
\end{aligned}$$

For ρ_3 we have

$$\begin{aligned}
\rho_3(M_\alpha,M_\delta,M_\beta) &= \frac{\sqrt{\mathrm{tr}(M_\delta)^2-4}}{\upsilon(M_\delta)}\\
&= \frac{\sqrt{\big(K_1^n(\delta)+K_2^{n-1}(\delta)\big)^2-4}}{\upsilon(M_\delta)}\\
&= \rho_4\big(\upsilon(M_\alpha),\upsilon(M_\delta),\upsilon(M_\beta)\big).
\end{aligned}$$

The result then follows from the previous three equalities. ∎

Proof of Proposition 3.4.2. Markov's theorem says that

$$\Psi_4\big(\mathcal{G}_\Sigma(1,5,2)\big)=\mathcal{M}_{<3}.$$

Combining this with Lemma 3.4.4 we have that

$$\begin{aligned}
\Psi_1\big(\mathcal{G}_{\oplus}\big((1,1),(2,2)\big)\big) &= \Psi_4 \circ X\big(\mathcal{G}_{\oplus}\big((1,1),(2,2)\big)\big) \\
&= \Psi_4\big(\mathcal{G}_{\Sigma}(1,5,2)\big) = \mathcal{M}_{<3}, \\
\Psi_2\big(\mathcal{G}_{\theta}\big(f_{(1,1)}, f_{(2,2)}\big)\big) &= \Psi_4 \circ \Lambda\big(\mathcal{G}_{\theta}\big(f_{(1,1)}, f_{(2,2)}\big)\big) \\
&= \Psi_4\big(\mathcal{G}_{\Sigma}(1,5,2)\big) = \mathcal{M}_{<3}, \\
\Psi_3\big(\mathcal{G}_{\bullet}\big(M_{(1,1)}, M_{(2,2)}\big)\big) &= \Psi_4 \circ \Upsilon\big(\mathcal{G}_{\bullet}\big(M_{(1,1)}, M_{(2,2)}\big)\big) \\
&= \Psi_4\big(\mathcal{G}_{\Sigma}(1,5,2)\big) = \mathcal{M}_{<3}
\end{aligned}$$

.

■

We relate the map Ψ_4 to the uniqueness conjecture with the following proposition.

Proposition 3.4.5. *If Ψ_4 is injective then the uniqueness conjecture for Markov numbers is true.*

Proof. If Ψ_4 is injective then there exists an inverse map

$$\Psi_4^{-1} : \mathcal{M}_{<3} \to \mathcal{G}_{\Sigma}(1,5,2),$$

taking single elements of $\mathcal{M}_{<3}$ to triples of Markov numbers. Hence the central element of each Markov vertex must uniquely define the triple. In other words, if (a,Q,b) and (c,Q,d) are Markov triples, then

$$\Psi_4(a,Q,b) = \frac{\sqrt{9Q^2-4}}{Q} = \Psi_4(c,Q,d),$$

and so

$$(a,Q,b) = \Psi_4^{-1}\left(\frac{\sqrt{9Q^2-4}}{Q}\right) = (c,Q,d).$$

Hence the triples are unique. ■

3.4.2 Perron identity for Markov sequences

We can now simplify the Perron identity for Markov sequences. We start by defining notation for the identity.

Definition 3.4.6. For a bi-infinite sequence of positive integers $\alpha = (a_i)_{-\infty}^{\infty}$ let

$$P(\alpha) = \sup_{i\in\mathbb{Z}} \left\{[a_i; a_{i+1} : \ldots] + [0; a_{i-1} : a_{i-2} : \ldots]\right\}.$$

For a finite sequence $\beta = (b_i)_1^m$ let

$$P(\beta) = \sup_{i=1,\ldots,m} \left\{[\langle b_i; \ldots : b_m : b_1 : \ldots : b_{i-1}\rangle] + [0; \langle b_{i-1} : \ldots : b_1 : b_m \ldots : b_i\rangle]\right\}.$$

By using a Markov sequence δ defined in the graph $\mathcal{G}_{\oplus}(\mu,\nu)$ we remove the need to find the supremum of the set

$$\big\{[a_i; a_{i+1} : \ldots] + [0; a_{i-1} : a_{i-1} : \ldots]\big\}$$

to work out the value of the Perron identity.

Recall the map ρ_1 acting on triples of Markov sequences (α,δ,β) by

$$\rho_1(\alpha,\delta,\beta) = \frac{\sqrt{\big(K_1^n(\delta) + K_2^{n-1}(\delta)\big)^2 - 4}}{K_1^{n-1}(\delta)},$$

where $\delta = (d_1,\ldots,d_n)$ and $n \geq 2$. We define an alternate map for triples of Markov sequences.

Definition 3.4.7. Define a map ρ_1^* taking triples of sequences at the vertices in $\mathcal{G}_{\oplus}(\mu,\nu)$ to $\mathbb{R}$ by the relation

$$\rho_1^*(\alpha,\delta,\beta) = \big[\langle\overline{\delta}\rangle\big] + [0; \langle\delta\rangle].$$

We show the equivalence of the maps ρ_1 and ρ_1^*.

Proposition 3.4.8. *For a triple of Markov sequences (α,δ,β) at a vertex of $\mathcal{G}_{\oplus}(\mu,\nu)$ we have that*

$$\rho_1(\alpha,\delta,\beta) = \rho_1^*(\alpha,\delta,\beta).$$

Proof. Let $\delta = (a_1,\ldots,a_n)$ with n even and $n \geq 1$. The value $\big[(\overline{\delta})\big]$ is the positive solution to the equation

$$K_1^{n-1}(\delta)x^2 + \big(K_2^{n-1}(\delta) - K_1^n(\delta)\big)x - K_2^n(\delta) = 0.$$

The value $[\langle\delta\rangle]$ is the positive solution to the equation

$$K_2^n(\delta)x^2 + \big(K_2^{n-1}(\delta) - K_1^n(\delta)\big)x - K_1^{n-1}(\delta) = 0.$$

Let

$$\begin{aligned}
\Delta(\delta) &= \big(K_1^n(\delta) + K_2^{n-1}(\delta)\big)^2 - 4,\\
b &= K_1^n(\delta) - K_2^{n-1}(\delta),\\
D &= 4K_2^n(\delta)K_1^{n-1}(\delta).
\end{aligned}$$

Then

$$\begin{aligned}
\rho_1^*(\alpha,\delta,\beta) &= \big[\langle\overline{\delta}\rangle\big] + [0; \langle\delta\rangle]\\
&= \frac{b + \sqrt{\Delta(\delta)}}{2K_1^{n-1}(\delta)} + \frac{2K_2^n(\delta)}{b + \sqrt{\Delta(\delta)}}\\
&= \left(\frac{b^2 + \Delta(\delta) + D + 2b\sqrt{\Delta(\delta)}}{2K_1^{n-1}(\delta)\Big(b + \sqrt{\Delta(\delta)}\Big)}\right)\left(\frac{b - \sqrt{\Delta(\delta)}}{b - \sqrt{\Delta(\delta)}}\right)\\
&= \frac{b\big(b^2 - \Delta(\delta) + D\big) + \big(b^2 - \Delta(\delta) - D\big)\sqrt{\Delta(\delta)}}{2K_1^{n-1}(\delta)\big(b^2 - \Delta(\delta)\big)}.
\end{aligned}$$

In the third equality we multiply by the value

$$\frac{b-\sqrt{\Delta(\delta)}}{b-\sqrt{\Delta(\delta)}}=1$$

to remove the square root in the denominator. Note that

$$\begin{aligned}
b^2-\Delta(\delta) &= \big(K_1^n(\delta)-K_2^{n-1}(\delta)\big)^2-\big(K_1^n(\delta)+K_2^{n-1}(\delta)\big)^2+4\\
&= -4K_1^n(\delta)K_2^{n-1}(\delta)+4\\
&= -D,
\end{aligned}$$

since n is even. Hence

$$\begin{aligned}
\rho_1^*(\alpha,\delta,\beta) &= \frac{b(-D+D)+(-D-D)\sqrt{\Delta(\delta)}}{2K_1^{n-1}(\delta)(-D)}\\
&= \frac{-2D\sqrt{\Delta(\delta)}}{-2DK_1^{n-1}(\delta)}\\
&= \frac{\sqrt{\Delta(\delta)}}{K_1^{n-1}(\delta)}\\
&= \rho_1(\alpha,\delta,\beta).
\end{aligned}$$

This completes the proof. ∎

From this proposition and the result in Proposition 3.4.2 we have the proof of the following proposition.

Proposition 3.4.9. *For a sequence δ in any vertex of $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$ we have that*

$$P(\delta)=\big[\langle\overline{\delta}\rangle\big]+[0;\langle\delta\rangle].$$

We finish this subsection by collecting the results of Propositions 3.2.7, 3.2.20, 3.2.31, 3.3.7, 3.3.14, and 3.4.2, and Corollaries 3.2.36 and 3.2.41 together in the following theorem.

Theorem 3.4.10. *We have that*

$$\begin{aligned}
\Gamma\big(\mathcal{G}_\oplus(\mu,\nu)\big) &= \mathcal{G}_\bullet\big(M_\mu,M_\nu\big),\\
\Gamma^{-1}\big(\mathcal{G}_\bullet\big(M_\mu,M_\nu\big)\big) &= \mathcal{G}_\oplus(\mu,\nu),\\
\Phi\big(\mathcal{G}_\oplus(\mu,\nu)\big) &= \mathcal{G}_\theta\left(f_\mu,f_\nu\right),\\
\Phi^{-1}\big(\mathcal{G}_\theta\big(f_\mu,f_\nu\big)\big) &= \mathcal{G}_\oplus(\mu,\nu),\\
\Omega\big(\mathcal{G}_\bullet\big(M_\mu,M_\nu\big)\big) &= \mathcal{G}_\theta\big(f_\mu,f_\nu\big),\\
\Omega^{-1}\big(\mathcal{G}_\theta\big(f_\mu,f_\nu\big)\big) &= \mathcal{G}_\bullet\big(M_\mu,M_\nu\big),
\end{aligned}
\qquad
\begin{aligned}
X\Big(\mathcal{G}_\oplus\big((1,1),(2,2)\big)\Big) &= \mathcal{G}_\Sigma(1,5,2),\\
\Lambda\big(\mathcal{G}_\theta\big(f_{(1,1)},f_{(2,2)}\big)\big) &= \mathcal{G}_\Sigma(1,5,2),\\
\Upsilon\big(\mathcal{G}_\bullet\big(M_{(1,1)},M_{(2,2)}\big)\big) &= G_\Sigma(1,5,2),\\
\Psi_1\big(\mathcal{G}_\oplus\big((1,1),(2,2)\big)\big) &= \mathcal{M}_{<3},\\
\Psi_2\big(\mathcal{G}_\theta\big(f_{(1,1)},f_{(2,2)}\big)\big) &= \mathcal{M}_{<3},\\
\Psi_3\big(\mathcal{G}_\bullet\big(M_{(1,1)},M_{(2,2)}\big)\big) &= \mathcal{M}_{<3},\\
\Psi_4\big(\mathcal{G}_\Sigma(1,5,2)\big) &= \mathcal{M}_{<3}.
\end{aligned}$$

3.4.3 Partial inverse of Markov number maps

Recall Proposition 2.5.3, that says that there is an inverse map to Λ if and only if the uniqueness conjecture holds. The graphs $\mathcal{G}_{\oplus}\big((1,1),(2,2)\big)$, $\mathcal{G}_{\theta}\big(f_{(1,1)},f_{(2,2)}\big)$, and $\mathcal{G}_{\bullet}\big(M_{(1,1)},M_{(2,2)}\big)$ are graph isomorphic. Hence the maps X and Υ have inverses if and only if the uniqueness conjecture holds.

In this short subsection we see that there are maps taking triples of Markov numbers (a,Q,b) to the regular Markov sequence, regular reduced Markov form, and regular reduced Markov matrix associated with the Markov number Q.

Statement 3.4.11 (Markov numbers to regular reduced Markov forms)**.** The definition of Markov forms in Markov's theorem (Theorem 1.1.9) defines a map taking triples (a,Q,b) in vertices of $\mathcal{G}_{\Sigma}(1,5,2)$ to the Markov form $f_{Q_{a,b}}(x,y)$. This may be described as a map from a graph to the set of Markov forms.

After an appropriate $\mathrm{SL}(2,\mathbb{Z})$ transformation (as defined in Proposition 3.2.33) we turn this Markov form $f_{Q_{a,b}}$ into a regular reduced Markov form, which we denote by f.

Statement 3.4.12 (Markov numbers to regular Markov sequences)**.** From Corollary 3.3.12 we know that the quadratic $f(x,1)$ decomposes into

$$f(x,1)=\breve{K}(\alpha)\big(x-[0;\langle\alpha\rangle]\big)\big(x+[\langle\overline{\alpha}\rangle]\big).$$

for some finite sequence of positive integers α. Then the regular Markov sequence of f is α.

Hence there exists a map taking a triple of regular Markov numbers (a,Q,b) to the regular Markov sequence associated with Q.

Statement 3.4.13 (Markov numbers to regular reduced Markov matrices)**.** From the regular Markov sequence α we get the regular Markov matrix M_α through direct computation. Hence we have a map taking a triple of regular Markov numbers (a,Q,b) to the regular reduced Markov matrix associated with Q.

Chapter 4

General Markov numbers

In this chapter we present an extension to the theory of Markov numbers. Various other extensions have been proposed. Many are based on variations of the Markov Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz.$$

In particular we mention the work of S. Perrine [74], and the subsequent work by T. Cusick [27]. These works deal with finite sequences arising from the study of solutions of generalised Diophantine equations

$$x^2 + y^2 + z^2 = 4xyz - nx,$$

for n a positive integer. A. Baragar [4, 6] studied solutions of the equation

$$x_1^2 + \ldots + x_n^2 = ax_1 \cdots x_n,$$

for positive integers a and n.

We take a different approach here. We define graphs of sequences from which we generate graphs of matrices, forms, and numbers, all of which share relations similar to the regular case in Chapter 3.

In Section 4.1 we define Markov graphs. These are graphs $\mathcal{G}_\oplus(\mu, \nu)$ that have the property that any form f in any vertex of $\Gamma\big(\mathcal{G}_\oplus(\mu, \nu)\big)$ attains its Markov minimum at the point $(1, 0)$.

To check whether a graph is Markov or not requires knowing about the Markov minima of all forms in $\Gamma\big(\mathcal{G}_\oplus(\mu, \nu)\big)$. We introduce conditions that allow us to easily check the Markov minima for certain sequences μ and ν in Theorems 4.1.14 and 4.1.27, introduced in the papers [55] and [94] respectively. Due to its length, we prove Theorem 4.1.27 in its own section, Section 4.2.

We define graphs of general reduced Markov forms, general reduced Markov matrices, and general Markov numbers in Section 4.3. We present Theorem 4.3.4, specifying all reduced matrices in the equivalence classes defined by periodic LLS sequences.

In Section 4.4 we relate the graphs of general reduced Markov matrices, forms, and general Markov sequences to the Markov spectrum.

4.1 General Markov theory

In this section we extend the theory of Markov numbers. We define triple graphs of sequences with similar properties to the graph of regular Markov sequences. We then use the maps between graphs defined in Chapter 3 to generate graphs of forms, matrices, and numbers.

As seen in Chapter 2, LLS sequences of forms are $\mathrm{SL}(2,\mathbb{Z})$ invariant, whereas forms and matrices are not. For this reason we base our extension of Markov numbers on sequences. We start in Subsection 4.1.1 by finding conditions on sequences μ and ν such that every form f in every vertex of the graph $\Gamma\big(\mathcal{G}_{\oplus}(\mu,\nu)\big)$ satisfies

$$m(f) = f(1,0).$$

We call such a graph $\mathcal{G}_{\oplus}(\mu,\nu)$ a Markov graph. We derive the necessary conditions in Theorem 4.1.14 and Theorem 4.1.27 in Subsections 4.1.2 and 4.1.3.

For sequences μ and ν, recall the definitions from Chapter 3 of the triple graphs $\mathcal{G}_{\oplus}(\mu,\nu)$, $\mathcal{G}_{\bullet}\big(M_\mu, M_\nu\big)$, and $\mathcal{G}_{\omega}\big(f_\mu, f_\nu\big)$. Recall also that the maps Γ, Φ, and Ω are bijections, with inverses Γ^{-1}, Φ^{-1}, and Ω^{-1}.

4.1.1 Markov graphs

In this subsection we define special graphs of sequences called Markov graphs. This definition generalises the properties of regular Markov sequences.

We need some precursory definitions.

Definition 4.1.1. Let α be a finite sequence of integers. We call the infinite sequence whose period is α

$$\alpha\alpha\alpha\ldots$$

the periodisation of α, and denote it by

$$\langle\alpha\rangle.$$

We define an ordering of sequences.

Definition 4.1.2. Let $\alpha = (a_i)_{i=1}^{\infty}$ and $\beta = (b_i)_{i=1}^{\infty}$ be infinite sequences of real numbers. We write

$$\alpha \succ \beta$$

if there is some positive integer n such that $a_i = b_i$ for $i = 1, \ldots, n-1$ and

$$\begin{cases} a_n > b_n & n \text{ is odd,} \\ a_n < b_n & n \text{ is even.} \end{cases}$$

Let the periodisation of finite sequences α and β be

$$\langle\alpha\rangle = (a_i)_{i=1}^{\infty},$$
$$\langle\beta\rangle = (b_i)_{i=1}^{\infty}.$$

We write

$$\langle\alpha\rangle \prec \langle\beta\rangle$$

if $(\alpha)_{i=1}^{\infty} \prec (\beta)_{i=1}^{\infty}$.

Definition 4.1.3. Let $\alpha = (\alpha_1, \ldots, \alpha_{2n})$.

(i) We call α *evenly palindromic* if there is some $i \in \{1, \ldots, 2n\}$ such that the cyclic shift α_i is palindromic, i.e.

$$\alpha_i = (\alpha_i, \ldots, \alpha_n, \alpha_1, \ldots, \alpha_{i-1}) = (\alpha_{i-1}, \ldots, \alpha_1, \alpha_n, \ldots, \alpha_i) = \overline{\alpha_i}.$$

(ii) If there exists some even sequence β such that

$$\alpha = \beta \oplus \beta \oplus \ldots \oplus \beta$$

then we call α *evenly composite*. If α is not evenly composite then we call it *evenly prime*.

Now we have the definition of Markov graphs.

Definition 4.1.4. Let μ and ν be evenly-prime sequences of integers. We call $\mathcal{G}_{\oplus}(\mu, \nu)$ an *almost Markov graph* if the following two conditions hold.

- $\langle\mu\rangle \prec \langle\nu\rangle$ and $\langle\overline{\mu}\rangle \prec \langle\overline{\nu}\rangle$.
- The forms f_μ and f_ν attain their Markov minima at $(1, 0)$.

We call $\mathcal{G}_{\oplus}(\mu, \nu)$ a *Markov graph* if the two previous conditions hold as well as the following.

- Every sequence in every triple of $\mathcal{G}_{\oplus}(\mu, \nu)$ is evenly palindromic.

Definition 4.1.5. Let $\mathcal{G}_{\oplus}(\mu, \nu)$ be an almost Markov graph. Let α be a sequence in any vertex of $\mathcal{G}_{\oplus}(\mu, \nu)$. Then we call α an *almost Markov sequence*.

Let $\mathcal{G}_{\oplus}(\mu, \nu)$ be a Markov graph. Let α be a sequence in any vertex of $\mathcal{G}_{\oplus}(\mu, \nu)$. Then we call α a *general Markov sequence*.

Definition 4.1.6. Let $\alpha = (a_1, \ldots, a_n)$ be a sequence in some vertex of a Markov graph $\mathcal{G}_{\oplus}(\mu, \nu)$. Recall the definition of a form associated with α as

$$f_\alpha(x, y) = K_1^{n-1}(\alpha)x^2 + \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)xy - K_2^n(\alpha)y^2.$$

We call the graph $\Phi\big(\mathcal{G}_{\oplus}(\mu, \nu)\big) = \mathcal{G}_\theta\big(f_\mu, f_\nu\big)$ a *graph of general reduced Markov forms*. We call any form f_α in any vertex of $\mathcal{G}_\theta\big(f_\mu, f_\nu\big)$ a *general reduced Markov form*.

Recall Proposition 2.2.34, that says the Markov minimum of a form given by a sequence α is the minimum of the set

$$\big\{\breve{K}(\alpha_i) \mid i = 0, \dots, |\alpha| - 1\big\},$$

where α_i is the i-th circular shift of α.

Example 45. We give some examples of almost Markov graphs. We look at the conditions for the following sequences.

$$\begin{array}{ll} \alpha_1 = (1,1) & \beta_1 = (1,1,2,2), \\ \alpha_2 = (a,a) & \beta_2 = (b,b), \quad \text{for } a < b \text{ positive integers}, \\ \alpha_3 = (4,4,4,1,2,2,1,4) & \beta_3 = (11,11,11,1,2,2,1,11), \\ \alpha_4 = (3,3,2,1,3,3) & \beta_4 = (5,5,1,2,5,5). \end{array}$$

For α_1 and β_1 we have for the extended sequences

$$\alpha_1^2 = (1,1,1,1) \prec (1,1,2,2) = \beta_1,$$
$$\overline{\alpha_1} = (1,1) \prec (2,2,1,1) = \overline{\beta_1}.$$

Also

$$\breve{K}(1,1) = 1, \quad \breve{K}(1,1,2,2) = 5, \quad \breve{K}(2,1,1,2) = 5,$$
$$\breve{K}(2,2,1,1) = 7, \quad \breve{K}(1,2,2,1) = 7.$$

Hence $\mathcal{G}_\oplus(\alpha_1, \beta_1)$ is an almost Markov graph.

The first two conditions are clear for α_2 and β_2, so $\mathcal{G}_\oplus(\alpha_2, \beta_2)$ is also an almost Markov graph. In fact, for any non positive integers m and n the sequences $(a,a)^m$ and $(b,b)^n$ also satisfy the conditions of an almost Markov graph.

We have $\alpha_3 \prec \beta_3$ and $\overline{\alpha_3} \prec \overline{\beta_3}$ since $4 < 11$. Let $\mathrm{MIN}(\alpha)$ be the set of numbers $\breve{K}(\alpha_i)$ for all circular shifts of α. Then

$$\mathrm{MIN}(\alpha_3) = \{839, 1309, 2495, 931\},$$
$$\mathrm{MIN}(\beta_3) = \{14384, 52024, 111800, 15176\}.$$

Since $\breve{K}(\alpha_3) = 839$ and $\breve{K}(\beta_3) = 14384$ we have that $\mathcal{G}_\oplus(\alpha_3, \beta_3)$ is an almost Markov graph.

Finally consider α_4 and β_4. Clearly $\alpha_4 \prec \beta_4$, but

$$\mathrm{MIN}(\alpha_4) = \{121, 119, 142, 251, 109, 122\},$$
$$\mathrm{MIN}(\beta_4) = \{472, 431, 1537, 836, 457, 471\},$$

while $\breve{K}(\alpha_4) = 122 > 109$ and $\breve{K}(\beta_4) = 471 > 431$, so $\mathcal{G}_\oplus(\alpha_4, \beta_4)$ is not an almost Markov graph.

Remark 4.1.7. Since $\mathcal{G}_\oplus(\alpha_4, \beta_4)$ is not an almost Markov graph it cannot be a Markov graph. We will see in Example 52 that while both $\mathcal{G}_\oplus(\alpha_1, \beta_1)$ and $\mathcal{G}_\oplus(\alpha_3, \beta_3)$ are almost Markov, they are not Markov graphs. However, $\mathcal{G}_\oplus(\alpha_2, \beta_2)$ is Markov.

Remark 4.1.8. Note that the first condition of Definition 4.1.4 is easy to check. For the remainder of this section we simplify the second and third conditions, leaving us only to check simple results for the constructing sequences in a graph.

4.1.2 Minimum condition

We show that the second condition of Definition 4.1.4 reduces to showing that forms f_μ and f_ν attain their Markov minima at $(1, 0)$. This result is in Theorem 4.1.14, and was proven in the paper by O. Karpenkov and M. van Son [55]. Recall that for every regular reduced Markov form f_α the minimal value is the associated Markov number $Q = \breve{K}(\alpha)$. Moreover, the minimal value of the form is attained at $(1, 0)$. We start by defining reduced forms, and relating them to reduced matrices. Recall a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathrm{SL}(2, \mathbb{Z})$ is reduced if $0 \leq a \leq c < d$. The form associated with M is

$$f_M(x, y) = cx^2 + (d - a) - by^2.$$

Since $ad - bc = 1$ we have

$$(a - d)^2 + 4bc = (a + d)^2 - 4,$$

and in this case we have by factoring that

$$\frac{f_M(x, y)}{-b} = \left(y - \left(\frac{d{-}a{-}\sqrt{(a{+}d)^2{-}4}}{2b}\right) x\right) \left(y - \left(\frac{d{-}a{+}\sqrt{(a{+}d)^2{-}4}}{2b}\right) x\right).$$

Definition 4.1.9. We call a form f *reduced* if

$$f(x, y) = c(y - r_1 x)(y - r_2 x)$$

for some real numbers $c \neq 0$, $r_1 > 1$, and $-1 < r_2 < 0$.

Proposition 4.1.10. *Let*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a reduced $\mathrm{SL}(2, \mathbb{Z})$ *matrix with* $a > 0$. *Then the form* f_M *is reduced.*

Proof. Let $A = d - a$ and $B = (a + d)^2 - 4$. Note that $A > 0$ and $B > 0$. We show that

$$\frac{A + \sqrt{B}}{2b} > 1, \tag{4.1}$$

and

$$-1 < \frac{A - \sqrt{B}}{2b} < 0. \tag{4.2}$$

For Equation (4.2) we show that $A^2 < B$. We have

$$A^2 + 4ad - 4 = a^2 + d^2 + 2ad - 4 = B.$$

Since $4ad - 4 > 0$ we have
$$\frac{A - \sqrt{B}}{2b} < 0.$$
By definition we have that $0 < a \leq c < d$ and $ad - bc = 1$. Hence $b > 0$. Also $a(d-c) > 1$. Hence

$$ac + c^2 < dc + ad - 1 = dc + bc.$$

From this we have that $d - a + b > c$, and so, since $b > 0$, we have

$$\begin{aligned}
b(d-a) + b^2 &> bc, \\
4b(d-a) + 4b^2 &> 4bc = 4ad - 4, \\
-2ad + 4b(d-a) + 4b^2 &> 2ad - 4, \\
a^2 + d^2 - 2ad + 4b(d-a) + 4b^2 &> a^2 + d^2 + 2ad - 4, \\
(d-a)^2 + 4b(d-a) + 4b^2 &> (a+d)^2 - 4, \\
(A - 2b)^2 &> \sqrt{B}^2.
\end{aligned}$$

From this we have
$$A - \sqrt{B} > -2b$$

and so Equation (4.2) holds.

For Equation (4.1) we show that

$$A + \sqrt{B} > 2b.$$

If $A > 2b$ then this holds, else we show that

$$\begin{aligned}
B > (2b - A)^2 &= 4b^2 + 4ba + a^2 + d^2 - 4bd - 2ad \\
a^2 + d^2 + 2ad - 4 &> 4b^2 + 4ba + a^2 + d^2 - 4bd - 2ad.
\end{aligned}$$

This inequality holds if and only if

$$\begin{aligned}
2ad + 4bd + 2ad &> 4 + 4b^2 + 4ba \\
ad + bd &> b^2 + ba + 1 \\
bc + 1 + bd &> b^2 + ba + 1 \\
c + d &> b + a.
\end{aligned}$$

Since $ad - bc = 1$ we have that this inequality is equivalent to

$$\begin{aligned}
c + d &> \frac{ad - 1}{c} + a \\
c^2 + cd &> ad + ac - 1.
\end{aligned}$$

This inequality holds since $c \geq a$.

■

Corollary 4.1.11. Let $\alpha = (a_1, \ldots, a_n)$ be a general Markov sequence. The matrix associated with α is

$$M_\alpha = \begin{pmatrix} K_2^{n-1}(\alpha) & K_2^n(\alpha) \\ K_1^{n-1}(\alpha) & K_1^n(\alpha) \end{pmatrix}.$$

The associated general reduced Markov form

$$f_\alpha(x,y) = K_1^{n-1}(\alpha)x^2 + \big(K_1^n(\alpha) - K_2^{n-1}(\alpha)\big)xy - K_2^n(\alpha)y^2$$

is reduced.

This follows since the matrix M is reduced with $\mathrm{tr}(M) = 1$ and $K_2^{n-1}(\alpha) > 0$.

Definition 4.1.12. Let a and b be real numbers. Let $g_{a,b}$ be the form defined by the equation

$$g_{a,b} = (x - ay)(x - by).$$

Denote the cone defined by $x - ay > 0$ and $x - by > 0$ by

$$U_{+,+}^{a,b}.$$

Denote the cone defined by $x - ay < 0$ and $x - by > 0$ by

$$U_{-,+}^{a,b}.$$

We use the following notation. For a finite sequence μ we denote the pair of infinite continued fractions $[0; \langle\mu\rangle]$ and $-[\langle\overline{\mu}\rangle]$ by

$$[\langle\langle\mu\rangle\rangle] = [0; \langle\mu\rangle], -[\langle\overline{\mu}\rangle]\,.$$

Example 46. Figure 4.1 shows the sails and cones of

$$\mu = (1,1) \text{ and } \mu\nu = (1,1,10,10)$$

for $y > 0$. This gives a visualisation of $U_{+,+}^{[\langle\langle\mu\rangle\rangle]}$ and $U_{+,+}^{[\langle\langle\mu\nu\rangle\rangle]}$ for the forms $g_{[\langle\langle\nu\rangle\rangle]}$ and $g_{[\langle\langle\mu\nu\rangle\rangle]}$.

Lemma 4.1.13. Note that if $\langle\mu\rangle \prec \langle\nu\rangle$ then

$$U_{+,+}^{[\langle\langle\mu\rangle\rangle]} \cap U_{+,+}^{[\langle\langle\mu\nu\rangle\rangle]} = U_{+,+}^{[\langle\langle\mu\rangle\rangle]}.$$

The following important theorem is proven by O. Karpenkov and M. van Son in [55].

Theorem 4.1.14. *Let μ and ν be two evenly-prime sequences of integers. Assume that*

- $\langle\mu\rangle \prec \langle\nu\rangle$ *and* $\langle\overline{\mu}\rangle \prec \langle\overline{\nu}\rangle$*;*
- *The global minimum of $g_{[\langle\langle\mu\rangle\rangle]}$ at non zero-integer points of the cone $U_{+,+}^{[\langle\langle\mu\rangle\rangle]}$ is attained at $(1,0)$.*
- *The global minimum of $g_{[\langle\langle\nu\rangle\rangle]}$ at non zero-integer points of the cone $U_{+,+}^{[\langle\langle\nu\rangle\rangle]}$ is attained at $(1,0)$.*

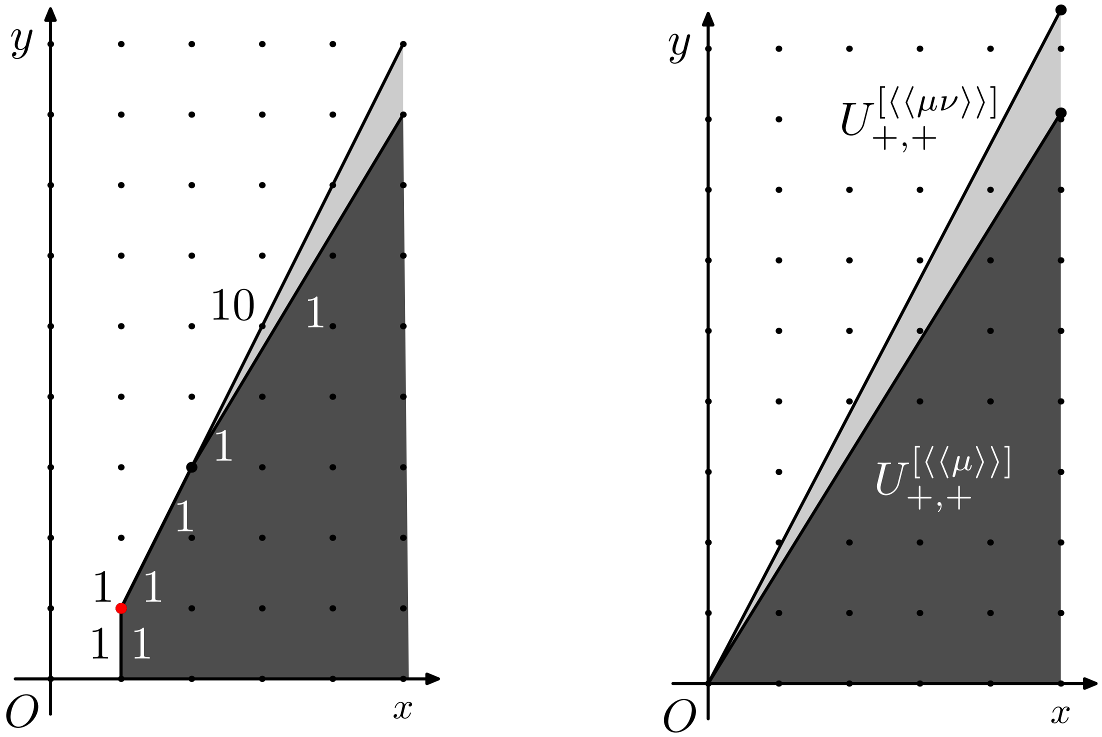

Figure 4.1: The left figure shows the sails for $\mu = (1, 1)$ and $\mu\nu = (1, 1, 10, 10)$. The right figure shows the corresponding cones for $y > 0$.

Then the global minimum of $g_{[\langle\langle\mu\nu\rangle\rangle]}$ at non-zero integer points of the cone $U_{+,+}^{[\langle\langle\mu\nu\rangle\rangle]}$ is attained at $(1, 0)$.

This global minimum is the unique global minimum on the sail for $g_{[\langle\langle\mu\nu\rangle\rangle]}$ up to lattice preserving translations of the sail.

To prove this theorem we need the following two lemmas and corollary, all proved in our joint paper with O. Karpenkov [55].

Lemma 4.1.15. Let a and b be real numbers. Let $v = (x, y)$ be a point in the region $U_{+,+}^{a,b}$, and let ε be a positive real number. Then the following inequalities hold.

(i) If $y > 0$ then
$$\begin{cases} g_{a+\varepsilon,b}(v) < g_{a,b}(v) \\ g_{a,b+\varepsilon}(v) < g_{a,b}(v) \end{cases}.$$

(ii) If $y < 0$ then
$$\begin{cases} g_{a+\varepsilon,b}(v) > g_{a,b}(v) \\ g_{a,b+\varepsilon}(v) > g_{a,b}(v) \end{cases}.$$

(iii) If $y = 0$ then
$$\begin{cases} g_{a\pm\varepsilon,b}(v) = g_{a,b}(v) \\ g_{a,b\pm\varepsilon}(v) = g_{a,b}(v) \end{cases}.$$

Proof. Consider $v = (x, y) \in U^{a,b}_{+,+}$, and $\varepsilon > 0$. We have

$$g_{a+\varepsilon,b}(x, y) - g_{a,b}(x, y) = -\varepsilon y(x - by).$$

The sign of this expression is opposite to the sign of y since $x - by > 0$ for every point of the region $U^{a,b}_{+,+}$ and $\varepsilon > 0$. Therefore if $y > 0$ then the difference is negative, and hence $g_{a+\varepsilon,b}(v) < g_{a,b}(v)$.

All the other three cases of (*i*) and (*ii*) are analogous.

Item (*iii*) is trivial. ∎

Lemma 4.1.16. Let a_1, a_2, b_1, and b_3 be the real numbers defined by the infinite continued fractions

$$\begin{aligned} a_1 &= [0; p_1 : p_2 : \ldots], \\ a_2 &= [0; q_1 : q_2 : \ldots], \\ b_1 &= -[r_1 : r_2 : \ldots], \\ b_2 &= -[s_1 : s_2 : \ldots]. \end{aligned}$$

Let $(p)^\infty_{n=1} \prec (q)^\infty_{n=1}$ and $(r)^\infty_{n=1} \prec (s)^\infty_{n=1}$. Let $v = (x, y)$ be a point in the intersection

$$v \in U^{a_1,b_1}_{+,+} \cap U^{a_2,b_2}_{+,+}.$$

Then the following equality holds:

$$\operatorname{sign}\big(g_{a_1,b_1}(v) - g_{a_2,b_2}(v)\big) = \operatorname{sign}(y).$$

Proof. Since $(p)^\infty_{n=1} \prec (q)^\infty_{n=1}$ we have that $a_1 > a_2$. Also, since $(r)^\infty_{n=1} \prec (s)^\infty_{n=1}$ we have that $b_1 > b_2$. Then the proposition follows as a direct corollary of Lemma 4.1.15 applied twice. ∎

Corollary 4.1.17. Let μ and ν be evenly-prime sequences of integers. Assume that $\langle\mu\rangle \prec \langle\nu\rangle$ and $\langle\overline{\mu}\rangle \prec \langle\overline{\nu}\rangle$.

(a) Consider a point v in the cone $U^{[\langle\langle\mu\rangle\rangle]}_{+,+}$. Note that

$$v \in U^{[\langle\langle\mu\rangle\rangle]}_{+,+} \cap U^{[\langle\langle\mu\nu\rangle\rangle]}_{+,+} \quad \Big(= U^{[\langle\langle\mu\rangle\rangle]}_{+,+}\Big),$$

by Lemma 4.1.13. Then

$$\operatorname{sign}\big(g_{[\langle\langle\mu\nu\rangle\rangle]}(v) - g_{[\langle\langle\mu\rangle\rangle]}(v)\big) = \operatorname{sign}(y).$$

(b) Consider a point v in the cone $U^{[\langle\langle\mu\nu\rangle\rangle]}_{+,+}$. Note that

$$v \in U^{[\langle\langle\nu\rangle\rangle]}_{+,+} \cap U^{[\langle\langle\mu\nu\rangle\rangle]}_{+,+} \quad \Big(= U^{[\langle\langle\mu\nu\rangle\rangle]}_{+,+}\Big),$$

by Lemma 4.1.13. Then

$$\operatorname{sign}\big(g_{[\langle\langle\mu\nu\rangle\rangle]}(v) - g_{[\langle\langle\nu\rangle\rangle]}(v)\big) = -\operatorname{sign}(y).$$

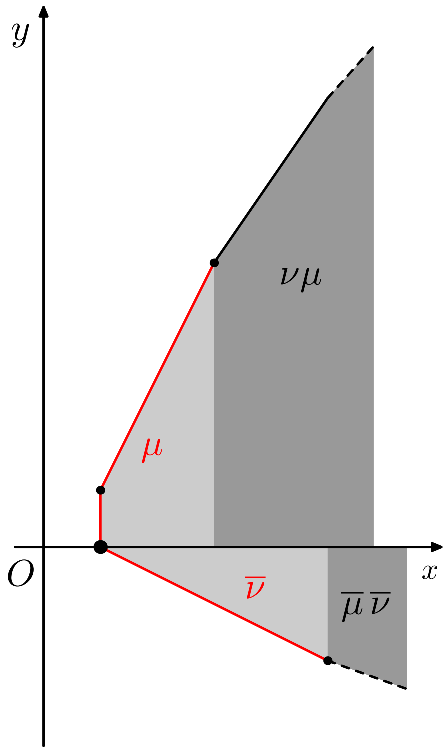

Figure 4.2: Stitching the sails of μ and ν together to get the sail of $\mu\nu$.

This corollary follows directly from Lemma 4.1.16. Now we prove Theorem 4.1.14.

Proof of Theorem 4.1.14. We show that the minimum of the form $g_{[\langle\langle\mu\nu\rangle\rangle]}$ in the cone $U_{+,+}^{[\langle\langle\mu\nu\rangle\rangle]}$ is attained at $(1, 0)$.

1. Consider the sail of $g_{[\langle\langle\mu\nu\rangle\rangle]}$ in the region $U_{+,+}^{[\langle\langle\mu\nu\rangle\rangle]}$. Note that $(1, 0)$ is a vertex of this sail. Note also that every period of this sail contains at least one vertex at which $g_{[\langle\langle\mu\nu\rangle\rangle]}$ is minimal (by Corollary 2.2.30).

2. The LLS sequence of this sail is made up of repeating periods $\mu\nu$ in the $y > 0$ region, and $\overline{\nu\mu}$ in the $y < 0$ region, which intersect at the vertex $(1, 0)$. We consider the period of the sail containing the point $(1, 0)$ as a vertex, and with LLS sequence $\overline{\nu}\mu$ as illustrated in Figure 4.2. Denote this period of the sail by $[\overline{\nu}, \mu]$.

3. Let $(x, y) \neq (1, 0)$ be a vertex on the period of the sail $[\overline{\nu}, \mu]$ with $y > 0$. Then

$$g_{[\langle\langle\mu\nu\rangle\rangle]}(1, 0) = g_{[\langle\langle\mu\rangle\rangle]}(1, 0) < g_{[\langle\langle\mu\rangle\rangle]}(x, y) < g_{[\langle\langle\mu\nu\rangle\rangle]}(x, y).$$

   The first equality holds by definition of the forms g. The second inequality holds by the assumption that the global minimum of $g_{[\langle\langle\mu\rangle\rangle]}(1, 0)$ in the cone $U_{+,+}^{[\langle\langle\mu\rangle\rangle]}$ is attained at $(1, 0)$. The third inequality holds by Corollary 4.1.17.

4. By similar arguments we have for a vertex $(x, y) \neq (1, 0)$ on the period of the sail $[\overline{\nu}, \mu]$ with $y < 0$ that

$$g_{[\langle\langle\mu\nu\rangle\rangle]}(1, 0) = g_{[\langle\langle\nu\rangle\rangle]}(1, 0) < g_{[\langle\langle\nu\rangle\rangle]}(x, y) < g_{[\langle\langle\mu\nu\rangle\rangle]}(x, y).$$

Hence the global minimum of $g_{[\langle\langle\mu\nu\rangle\rangle]}$ in the cone $U_{+,+}^{[\langle\langle\mu\nu\rangle\rangle]}$ is attained at $(1, 0)$.

■

Corollary 4.1.18. The graph $\mathcal{G}_{\oplus}\big((1, 1), (2, 2)\big)$ is almost Markov.

Example 47. Let $\alpha = (1, 1, 2, 3)$ and $\beta = (7, 4)$. Clearly $\alpha \prec \beta$ and $\overline{\alpha} \prec \overline{\beta}$. The forms f_α, f_β, and $f_{\alpha\beta}$ are given by

$$\begin{aligned} f_\alpha(x, y) &= 5x^2 + 14xy - 10y^2, \\ f_\beta(x, y) &= 7x^2 + 28xy - 4y^2, \\ f_{\alpha\beta}(x, y) &= 124x^2 + 440xy - 302y^2. \end{aligned}$$

The values that the form f_α attains at the vertices of its sail that contains the vertex $(1, 0)$ are

$$\big\{\breve{K}(\alpha_i) \mid i \text{ odd}\big\} = \{5, 9\},$$

and the values attained at the vertices of the dual sail are

$$\big\{\breve{K}(\alpha_i) \mid i \text{ even}\big\} = \{-10, -7\}.$$

Hence the minimal value that f_α attains at integer points in the cone $U_{+,+}^{[\langle\langle\alpha\rangle\rangle]}$ is

$$f_\alpha(1, 0) = 5.$$

Similarly, the minimal value that f_β attains at integer points in the cone $U_{+,+}^{[\langle\langle\beta\rangle\rangle]}$ is

$$f_\beta(1, 0) = 7.$$

Theorem 4.1.14 says that the minimal value the form $f_{\alpha\beta}$ attains at integer points in the cone $U_{+,+}^{[\langle\langle\alpha\beta\rangle\rangle]}$ is

$$f_{\alpha\beta}(1, 0) = 124.$$

Indeed, we have that

$$\Big\{\breve{K}\big((\alpha\beta)_i\big) \mid i \text{ odd}\Big\} = \{124, 262, 166\}.$$

Note however that the Markov minima of f_β is not attained at the point $(1, 0)$, or at any integer point in the cone $U_{+,+}^{[\langle\langle\beta\rangle\rangle]}$. Similarly, the Markov minima of $f_{\alpha\beta}$ is not attained at $(1, 0)$ either. In fact we have

$$m\big(f_{\alpha\beta}\big) = |f_{\alpha\beta}(-4, 1)| = 78.$$

The question of when minimal value that a form f_α can attain at integer points of the cone $U_{+,+}^{[\langle\langle\alpha\rangle\rangle]}$ is different to the maximal value attained at integer points of the cone $U_{-,+}^{[\langle\langle\alpha\rangle\rangle]}$ is the following.

Question. Do there exist sequences α and β satisfying $\alpha \prec \beta$ and $\overline{\alpha} \prec \overline{\beta}$ and

$$m\big(f_\alpha\big) = f_\alpha(1,0),$$
$$m\big(f_\beta\big) = f_\beta(1,0),$$

such that

$$\min\big\{\breve{K}\big((\alpha\beta)_i\big) \mid i \text{ even}\big\} < \min\big\{\breve{K}\big((\alpha\beta)_i\big) \mid i \text{ odd}\big\}?$$

Clearly, if a sequence α is evenly palindromic then its form's Markov minimum $m\big(f_\alpha\big)$ is attained at some vertex of the sails in both cones $U_{+,+}^{[\langle\langle\alpha\rangle\rangle]}$ and $U_{-,+}^{[\langle\langle\alpha\rangle\rangle]}$. Whether an almost Markov sequence's form f_β can attain its Markov minimum only at a vertex of the sail in the cone $U_{-,+}^{[\langle\langle\alpha\rangle\rangle]}$ is unknown to the author.

Theorem 4.1.14 says that the forms associated with the sequences in an almost Markov graph have minimal positive value at $(1,0)$. We present the following corollary.

Corollary 4.1.19. Let μ and ν be two evenly-prime sequences of integers. Assume that

- $\langle\mu\rangle \prec \langle\nu\rangle$ and $\langle\overline{\mu}\rangle \prec \langle\overline{\nu}\rangle$;
- The global minimum of $g_{[\langle\langle\mu\rangle\rangle]}$ at non zero integer points of the cone $U_{+,+}^{[\langle\langle\mu\rangle\rangle]}$ is attained at $(1,0)$.
- The global minimum of $g_{[\langle\langle\nu\rangle\rangle]}$ at non zero integer points of the cone $U_{+,+}^{[\langle\langle\nu\rangle\rangle]}$ is attained at $(1,0)$.
- The sequence $\langle\mu\nu\rangle$ is evenly palindromic.

Then the global minimum of $|g_{[\langle\langle\mu\nu\rangle\rangle]}|$ at non zero integer points is attained at $(1,0)$. This global minimum is the unique global minimum on the sail for $g_{[\langle\langle\mu\nu\rangle\rangle]}$ up to lattice preserving translations of the sail.

Remark 4.1.20. Theorem 4.1.14 says that we only need the forms f_μ and f_ν to be minimal at $(1,0)$, and $\langle\mu\rangle \prec \langle\nu\rangle$ and $\langle\overline{\mu}\rangle \prec \langle\overline{\nu}\rangle$, for every form f_α in every vertex of $\Phi\big(\mathcal{G}_\oplus(\mu,\nu)\big)$ to have a minimum at $(1,0)$ in the cone $U_{+,+}^{[\langle\langle\alpha\rangle\rangle]}$. This simplifies the task of checking whether a graph of sequences is almost Markov or not.

Example 48. We give a simple example to illustrate the result in Theorem 4.1.14. Let $\alpha = (5,3,3,5)$ and $\beta = (7,1,1,7)$. Then $\mathcal{G}_\oplus(\mu,\nu)$ is an almost Markov graph (i.e. it satisfies the conditions of Theorem 4.1.14). Consider the sequence $\delta = \alpha\alpha\beta\alpha\beta$, which is contained in a vertex of $\mathcal{G}_\oplus(\mu,\nu)$. The set of values $\breve{K}(\delta_i)$ for all circular shifts of δ, denoted by $\mathrm{MIN}(\delta)$, is

$$\begin{aligned}\mathrm{MIN}(\delta) = \{&41950502695, 41950506331, 58994258149,\\ &59600592055, 59600797489, 92332302595,\\ &92337714781, 92392956529, 160588595591, 160588597409\}.\end{aligned}$$

The minimal value of this set coincides with $\breve{K}(\delta) = 41950502695$, as the theorem states.

4.1.3 Evenly-palindromic Markov graphs

The final condition of Definition 4.1.4 for a graph $\mathcal{G}_\oplus(\mu,\nu)$ to be Markov is that each sequence in each vertex of the graph must be evenly palindromic. We show in Theorem 4.1.27 that this condition is fulfilled whenever the sequences μ and ν are palindromic.

We give an alternate definition for the sequences in the graph $\mathcal{G}_\oplus(\mu,\nu)$. First we need the following ordering of vertices of the graph.

Definition 4.1.21. Let w_1 and w_2 be two vertices in a graph $\mathcal{G}(S,\sigma,v)$ with

$$w_1 = \mathcal{L}_\sigma^{\alpha_{2n}}\mathcal{R}_\sigma^{\alpha_{2n-1}}\dots\mathcal{L}_\sigma^{\alpha_2}\mathcal{R}_\sigma^{\alpha_1}(v),$$
$$w_2 = \mathcal{L}_\sigma^{\beta_{2m}}\mathcal{R}_\sigma^{\beta_{2m-1}}\dots\mathcal{L}_\sigma^{\beta_2}\mathcal{R}_\sigma^{\beta_1}(v),$$

for positive integers n, m, $\alpha_1,\dots,\alpha_{2n}$, $\beta_1,\dots,\beta_{2m}$ satisfying

$$\sum_{i=1}^{2n}\alpha_i = \sum_{i=1}^{2m}\beta_i.$$

We call $\mathcal{L}_\sigma^{\alpha_{2n}}\mathcal{R}_\sigma^{\alpha_{2n-1}}\dots\mathcal{L}_\sigma^{\alpha_2}\mathcal{R}_\sigma^{\alpha_1}$ the *Farey sequence of* w_1. Define an ordering of vertices by

$$w_1 \prec_v w_2$$

if either

$$\begin{aligned} &\alpha_i = \beta_i, \text{ for } i = 1,\dots,2k{-}1, \ k < m,n \text{ and } \beta_{2k} < \alpha_{2k}, \text{ or}\\ &\alpha_i = \beta_i, \text{ for } i = 1,\dots,2k, \qquad k < m,n \text{ and } \alpha_{2k+1} < \beta_{2k+1}.\end{aligned}$$

Definition 4.1.22. Let *an ordered graph* be the pair $(\mathcal{G}(S,\sigma,v),\prec_v)$ be the graph $\mathcal{G}(S,\sigma,v)$ where each level n is ordered

$$w_1 \prec_v w_2 \prec_v \dots \prec_v w_{2^n}.$$

Example 49. Figure 4.3 shows the ordering of the first four levels of a triple graph. The vertices are labelled with pairs of numbers, the number in black denoting the level, and the number in red denoting the order of the vertex within the level

In an ordered graph $\mathcal{G}(S,\sigma,v)$ we have $\mathcal{L}_\sigma^1(v) \prec_v \mathcal{R}_\sigma^1(v)$. In the ordered graph $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$, for the vertices

$$w_1 = \Big((1,1),(1,1,1,1,1,1,2,2),(1,1,1,1,2,2)\Big),$$
$$w_2 = \Big((1,1,2,2),(1,1,2,2,1,1,2,2,2,2),(1,1,2,2,2,2)\Big),$$

we have that $w_1 = \mathcal{L}_\oplus^2(v)\mathcal{R}_\oplus^0(v)$ and $w_2 = \mathcal{L}_\oplus^1(v)\mathcal{R}_\oplus^1(v)$. Hence $w_1 \prec_v w_2$.

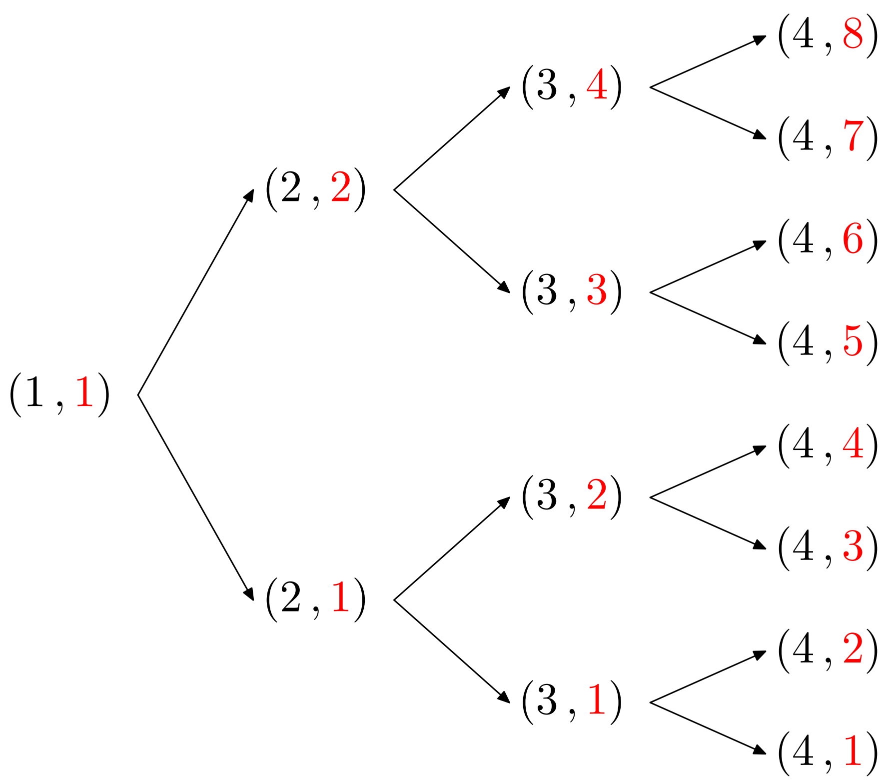

Figure 4.3: A triple graph with the level of the vertices written in black, and ordering within the level in red.

Remark 4.1.23. Here and below we assume that any graph $\mathcal{G}(S, \sigma, v)$ has the ordering of Definition 4.1.22.

Definition 4.1.24. For two sequences μ and ν define the function $S_n(\mu, \nu)$ by

$$S_0(\mu, \nu) = \mu, \quad S_1(\mu, \nu) = \nu, \quad S_2(\mu, \nu) = \mu \oplus \nu,$$

and for $n > 1$ and $1 \leq i \leq 2^{n-1}$ let $S_{2^{n-1}+i}(\mu, \nu)$ be the central element of the i-th vertex in the n-th level of the ordered graph $\big(\mathcal{G}_{\oplus}(\mu, \nu), \prec_v\big)$. We call $\big(S_i(\mu, \nu)\big)_{i \geq 0}$ the *ordered Markov sequences for μ and ν.*

Next we define Stern's diatomic sequence.

Definition 4.1.25. Let $d_0 = 0$ and $d_1 = 1$, and for all positive integers $n > 1$ let

$$d_{2n} = d_n, \qquad d_{2n-1} = d_n + d_{n-1}.$$

The sequence $(d_n)_{n \geq 0}$ is called *Stern's diatomic sequence.*

Example 50. Let $n = 15$. Then

$$S_n\big((1, 1), (2, 2)\big) = (1, 1, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2, 2).$$

Also

$$d_n = d_8 + d_7 = 1 + d_3 + d_4 = 2 + d_1 + d_2 = 4.$$

Table 4.1: The first 17 regular Markov sequences with d_n.

$n=1$	$(1,1)$ $d_0=0$	$(1,1,2,2)$ $d_1=1$	$(2,2)$ $d_2=1$
$n=2$	$(1,1,1,1,2,2)$ $d_3=2$	$(1,1,2,2,2,2)$ $d_4=1$	
$n=3$	$(1,1,1,1,1,1,2,2)$ $d_5=4$ $(1,1,2,2,2,2,2,2)$ $d_8=1$	$(1,1)(1,1,2,2)^2$ $d_6=2$	$(1,1,2,2)^2(2,2)$ $d_7=3$
$n=4$	$(1,1)^4(2,2)$ $d_9=4$ $(1,1)(1,1,2,2)^3$ $d_{12}=2$ $(1,1,2,2,2,2)^2(2,2)$ $d_{15}=4$	$(1,1)(1,1,1,1,2,2)^2$ $d_{10}=3$ $(1,1,2,2)^3(2,2)$ $d_{13}=5$ $(1,1)(2,2)^4$ $d_{16}=1$	$(1,1,1,1,2,2)^2(1,1,2,2)$ $d_{11}=5$ $(1,1,2,2)(1,1,2,2,2,2)^2$ $d_{14}=3$

Definition 4.1.26. Let $\Lambda = (\lambda_1, \dots, \lambda_m)$ for some positive m. Define the half sequences $\lfloor\Lambda\rfloor$ and $\lceil\Lambda\rceil$ by

$$\lfloor\Lambda\rfloor = (\lambda_1, \dots, \lambda_{\lfloor m/2\rfloor}) \quad \text{and} \quad \lceil\Lambda\rceil = (\lambda_{\lceil m/2\rceil}, \dots, \lambda_m).$$

We present a theorem that shows if two sequences μ and ν with even length are palindromic, then every sequence in every vertex of a graph $\mathcal{G}_\oplus(\mu,\nu)$ is evenly palindromic.

Theorem 4.1.27. *Let μ and ν be two palindromic sequences of positive integers. Let n be a positive integer. Let $\Lambda_i \in \{\mu,\nu\}$ such that*

$$S_n(\mu,\nu) = \Lambda_1 \dots \Lambda_N,$$

where N is a positive integer. Then the following sequences are palindromic

$$\Lambda_2 \dots \Lambda_{N-1}, \ \textit{and}$$
$$\begin{cases} \Lambda_{d_n/2+1}\Lambda_{d_n/2+2}\dots\Lambda_N\Lambda_1\dots\Lambda_{d_n/2}, & d_n \textit{ even},\\ \lceil\Lambda_{\lceil d_n/2\rceil}\rceil\Lambda_{\lceil d_n/2\rceil+1}\dots\Lambda_N\Lambda_1\dots\Lambda_{\lceil d_n/2\rceil-1}\lfloor\Lambda_{\lceil d_n/2\rceil}\rfloor, & d_n \textit{ odd}.\end{cases}$$

We postpone the proof of this theorem until Section 4.2.

Corollary 4.1.28. The graph $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$ is Markov.

Example 51. Table 4.1 compares $S_n\big((1,1),(2,2)\big)$ and d_n for $n=1,\dots,16$. The sequences are arranged by their levels in the graph $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$, and only the central sequences of each vertex are included, not the vertices themselves (except for the first level, where all sequences are shown).

According to Theorem 4.1.27, each sequence $S_n\big((1,1),(2,2)\big)$ will be palindromic after d_n circular shifts, since the sequences $(1,1)$ and $(2,2)$ both have length 2. Take for example the sequences

$$\alpha = S_7\big((1,1),(2,2)\big) \quad \text{and} \quad \beta = S_{11}\big((1,1),(2,2)\big).$$

Applying $d_7 = 3$ and $d_{11} = 5$ circular shifts we get

$$\begin{aligned} \alpha &= (1,1,2,2,1,1,2,2,2,2,2), \\ \alpha_3 &= (2,1,1,2,2,2,2,2,1,1,2), \end{aligned}$$

which is palindromic, and

$$\begin{aligned} \beta &= (1,1,1,1,2,2,1,1,1,1,2,2,1,1,2,2), \\ \beta_5 &= (2,1,1,1,1,2,2,1,1,2,2,1,1,1,1,2), \end{aligned}$$

which is also palindromic.

Example 52. Recall the sequences from Example 45:

$$\begin{aligned} &\alpha_1 = (1,1), && \beta_1 = (1,1,2,2), \\ &\alpha_2 = (a,a), && \beta_2 = (b,b), \quad \text{for } a < b \in \mathbb{Z}_+, \\ &\alpha_3 = (4,4,4,1,2,2,1,4), && \beta_3 = (11,11,11,1,2,2,1,11), \\ &\alpha_4 = (3,3,2,1,3,3), && \beta_4 = (5,5,1,2,5,5). \end{aligned}$$

The graph $\mathcal{G}_\oplus(\alpha_2,\beta_2)$ is a Markov graph since both α_2 and β_2 are palindromic.

The graph $\mathcal{G}_\oplus(\alpha_1,\beta_1)$ is not a Markov graph as β_1 is not palindromic. However we can take a circular shift of β_1 which is palindromic, and also has the same value as $\breve{K}(\beta_1)$. Hence $\mathcal{G}_\oplus\big((1,1),(2,1,1,2)\big)$ is a Markov graph.

Although we can take cyclic shifts of both α_3 and β_3 that are palindromic, these cyclic shifts do not give the Markov minima. Hence we cannot make a Markov graph from any circular shifts of α_3 or β_3.

Remark 4.1.29. Theorem 4.1.14 and Theorem 4.1.27 together allow us to create Markov graphs $\mathcal{G}_\oplus(\mu,\nu)$ by placing simple conditions on the sequences μ and ν.

4.2 Proof of Theorem 4.1.27

Palindromic Markov sequences

In this section we prove Theorem 4.1.27. This will be an expanded version of the proof found in the paper by the author [94].

The theorem is a statement on Markov sequences. A Markov sequence in a Markov graph $\mathcal{G}_{\oplus}(\mu, \nu)$ may be thought of as a finite word over an alphabet $\{\mu, \nu\}$. For instance $S_6(\mu, \nu) = \mu\mu\nu\mu\nu$. Words of this type were first studied by E. B. Christoffel [19] in 1873, and so are sometimes referred to as *Christoffel words.*

These words were related to Markov numbers by H. Cohn [20, 22], and also A. Glen et al. [41]. The combinatorics of these words have been studied in the following papers [34, 8, 11, 12, 9, 79]. Another resource on Christoffel words is by C. Kassell and C. Reutenauer [56].

The proof of Markov's theorem (Theorem 1.1.9) involves infinite analogues of Markov sequences called Sturmian words. These words have been studied in relation to Markov sequences, palindromic sequences, and the uniqueness conjecture [35, 78, 40, 13].

We show in Theorem 4.1.27 that Markov sequences are palindromic after a number of cyclic shifts.

Example 53. For a Markov graph $\mathcal{G}_{\oplus}(\alpha, \beta)$ we have that the cyclic shifts of the Markov sequences

$$\begin{aligned} S_6(\alpha, \beta) &= \alpha\alpha\beta\alpha\beta \\ &\quad\ \alpha\beta\alpha\beta\alpha \\ S_{15}(\alpha, \beta) &= \alpha\beta\beta\alpha\beta\beta\beta \\ &\quad\ \beta\alpha\beta\beta\beta\alpha\beta \end{aligned}$$

that are palindromic.

The exact number of cyclic shifts required to make Markov sequences palindromic is given by Stern's diatomic sequence. A useful exposition of Stern's diatomic sequence may be found in the paper by I. Urbiha [93].

The statement of the Theorem 4.1.27 appears new, but should not be too surprising given the frequency at which palindromes appear in the study of Markov numbers, and also that of the free group F_2. Palindromicity relating to the free group has been studied in the papers [73, 45, 7, 77]. Further work on palindromicity in relation to Markov numbers and the continued fraction expansion of quadratic irrationals may be found in [1, 66, 68, 67].

In the proof of our theorem we use sequences similar to Fibonacci sequences. We found these through sequence A022095 in [87]. Similar sequences appear in the works [65, 37, 33].

We start by recalling Definition 4.1.24. Let n and N be positive integers and let $A_i \in \{\mu, \nu\}$ for $i = 1, \dots, N$ be such that

$$S_n(\mu, \nu) = A_1 \dots A_N.$$

Remark 4.2.1. Part 1 of Theorem 4.1.27 says that the word $A_2 \dots A_{N-1}$ is palindromic. Define the word $\widehat{S_n(\mu, \nu)}$ by doubling every A_i in $S_n(\mu, \nu)$, so that

$$\widehat{S_n(\mu, \nu)} = A_1 A_1 A_2 A_2 \dots A_{N-1} A_{N-1} A_N A_N.$$

Then $\widehat{S_n(\mu, \nu)}$ is a sequence in some vertex of the graph $\mathcal{G}_\oplus(\mu\mu, \nu\nu)$. But this graph is the same as $\mathcal{G}_\oplus\big((\mu, \mu), (\nu, \nu)\big)$, whose sequences have the same structure as $\mathcal{G}_\oplus\big((1, 1), (2, 2)\big)$. Recall that for any regular Markov sequence $(b_1, \dots, b_m)$ we have $(b_3, \dots, b_{m-2}) = \overline{(b_{m-2}, \dots, b_3)}$. Hence we have that

$$A_2 \dots A_{N-1} = A_{N-1} \dots A_2.$$

In Subsection 4.2.1 we introduce Proposition 4.2.2, which contains a weaker version of the statement of Theorem 4.1.27 that we use in the final proof. We develop and prove Proposition 4.2.7, which gives us an alternate, and more useful, notation for the sequences in the Markov graph $\mathcal{G}_\oplus\big((1, 1), (2, 2)\big)$.

In Subsection 4.2.2 we state some technical lemmas for the proof of Proposition 4.2.2.

We develop the notational change of Subsection 4.2.1 further in Subsection 4.2.3. We show a construction of Markov sequences $S_n\big((1, 1), (2, 2)\big)$ in terms of powers $S_m\big((1, 1), (2, 2)\big)^j \oplus S_p\big((1, 1), (2, 2)\big)^k$, for some positive integers m and p less than n.

In Subsection 4.2.4 we prove Proposition 4.2.2.

Finally, we prove Theorem 4.1.27 in Subsection 4.2.5.

4.2.1 Alternative definition for $S(n)$

We first state Proposition 4.2.2 which serves as a basis for the proof of Theorem 4.1.27. Here and below we define the function S by $S(n) = S_n\big((1, 1), (2, 2)\big)$. We develop alternative definitions of the sequences $\big(S(n)\big)$ in Proposition 4.2.7.

We use the notation $C_i(\alpha)$ to denote the i-th cyclic shift of the sequence α.

Proposition 4.2.2. *Every sequence $S(i)$ is evenly palindromic. Moreover, we have that the sequence*

$$C_{d_i}\big(S(i)\big)$$

is palindromic, where d_i is the i-th element in Stern's diatomic sequence.

Example 54. $S(7) = (1,1,2,2,1,1,2,2,2,2,2)$, and $d_7 = 3$. Then

$$C_3\big(S(7)\big) = (2,1,1,2,2,2,2,1,1,2),$$

which is palindromic.

We define a sequence $\big(a(j)\big)_{j\geq 0}$ which we use to study the structure of the sequences $\big(S(j)\big)$.

Definition 4.2.3. Define a function a by $a(1){=}a(2){=}1$, and for all positive integers $j > 2$ by

$$a(2j) = a(j), \qquad \text{and} \qquad a(2j{-}1) = j.$$

The sequence $\big(a(j)\big)_{j\geq 0}$ is A003602 in [87]. Denote by $a^*(j)$ the function defined

$$a^*(j) = \begin{cases} a(j), & \text{if } a(j) > 1, \\ 0, & \text{if } a(j) = 1. \end{cases}$$

For the $a(j)$-th element in the sequence $\big(S(n)\big)$, we write $S\big(a(j)\big)$.

Example 55. The first 10 elements of $\big(a(j)\big)$ and (d_j) are given

$$\big(a(j)\big)_1^{10} = (1,1,2,1,3,2,4,1,5,3),$$
$$(d_j)_0^9 = (0,1,1,2,1,3,2,3,1,4).$$

Remark 4.2.4. Let v be the vertex in the n-th level of the ordered graph $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$ whose central element is $S(2^{n-1}{+}i)$. Note that since the graph is ordered we have that the central element of the vertex $\mathcal{L}_\oplus(v)$ is $S\big(2(2^{n-1}{+}i)-1\big)$, and the central element of $\mathcal{R}_\oplus(v)$ is $S\big(2(2^{n-1}{+}i)\big)$.

We have a proposition on the growth of the sequence $a(j)$.

Proposition 4.2.5. *For $j > 2$ we have that $2j > a(j)$.*

Proof. Assume j is odd, so $j = 2k{-}1$ for some positive integer $k > 1$. Then

$$a(j) = a(2k{-}1) = k = \frac{j+1}{2} < j < 2j.$$

Assume j is even. Then $j = 2^n(2k-1)$ for some positive integers $k < j$ and $n \geq 1$, and so

$$a(j) = a\big(2^n(2k{-}1)\big) = a(2k{-}1) = k < j < 2j.$$

Hence we have $a(j) < 2j$ for all $j > 2$. ■

Note that since $a(j) \leq a^*(j)$ and $a(j) < 2j$ we have that $a^*(j) < 2j{+}1$ and so $a^*(j{-}1) < 2j{-}1$. Now we define a function $\widehat{S}$ iteratively.

Definition 4.2.6. Define a function $\widehat{S}$ in the following way. Let

$$\widehat{S}(0) = (1,1), \quad \widehat{S}(1) = (2,2), \quad \widehat{S}(2) = (1,1,2,2).$$

For $j > 2$ let

$$\begin{aligned} \widehat{S}(2j) &= \widehat{S}(j) \oplus \widehat{S}\big(a(j)\big), \text{ and} \\ \widehat{S}(2j-1) &= \widehat{S}\big(a^*(j-1)\big) \oplus \widehat{S}(j). \end{aligned}$$

We relate the functions S and $\widehat{S}$ in the following proposition.

Proposition 4.2.7. *Let v be the vertex in the n-th level of the ordered graph $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$ whose central element is $S(2^n+i)$. If $i = 2k$ for some positive integer k, then*

$$v = \Big(\widehat{S}(2^{n-1}+k), S(2^n+i), \widehat{S}\big(a(2^{n-1}+k)\big)\Big).$$

If $i = 2k-1$ for some positive integer k, then

$$v = \Big(\widehat{S}\big(a^*(2^{n-1}+k-1)\big), S(2^n+i), \widehat{S}(2^{n-1}+k)\Big).$$

Remark 4.2.8. In particular, Proposition 4.2.7 says that $S(n) = \widehat{S}(n)$ for any $n \geq 0$.

Recall the map χ from Definition 3.2.6 that sends triples of sequences (α, δ, β) to triples of integers

$$\Big(\breve{K}(\alpha), \breve{K}(\delta), \breve{K}(\beta)\Big).$$

Example 56. Let $n = 42$. Then $a(21) = 10$ and so $\widehat{S}(42) = \widehat{S}(21) \oplus \widehat{S}(10)$. We show that $\chi\big(\widehat{S}(21), \widehat{S}(42), \widehat{S}(10)\big)$ is a vertex in $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$ by looking at the continuants of these sequences. We have

$$\begin{aligned} \widehat{S}(21) &= (1,1,1,1,2,2,1,1,1,1,2,2,1,1,1,1,2,2,1,1,2,2), \\ \widehat{S}(42) &= \widehat{S}(21) \oplus \widehat{S}(10), \\ \widehat{S}(10) &= (1,1,1,1,2,2,1,1,1,1,2,2,1,1,2,2). \end{aligned}$$

Let

$$\begin{aligned} x_1 &= \breve{K}\big(\widehat{S}(21)\big) = 294685, \\ x_2 &= \breve{K}\big(\widehat{S}(42)\big) = 6684339842, \\ x_3 &= \breve{K}\big(\widehat{S}(10)\big) = 7561. \end{aligned}$$

Then we have

$$x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3.$$

Hence $\chi\big(\widehat{S}(21), \widehat{S}(42), \widehat{S}(10)\big)$ is a vertex in $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$.

Example 57. Let $n = 37$. Then $a^*(18) = 5$ and so $\widehat{S}(37) = \widehat{S}(5) \oplus \widehat{S}(19)$. We show that $\chi\big(\widehat{S}(5), \widehat{S}(37), \widehat{S}(19)\big)$ is a vertex in $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$ by looking at the continuants of these sequences. We have

$$\begin{aligned}
\widehat{S}(5) &= (1,1,1,1,1,1,2,2),\\
\widehat{S}(37) &= (1,1,1,1,1,1,2,2,1,1,1,1,1,1,2,2,1,1,1,1,1,1,2,2,1,1,1,1,2,2),\\
\widehat{S}(19) &= (1,1,1,1,1,1,2,2,1,1,1,1,1,1,2,2,1,1,1,1,2,2).
\end{aligned}$$

Let

$$\begin{aligned}
y_1 &= \breve{K}\big(\widehat{S}(5)\big) = 34,\\
y_2 &= \breve{K}\big(\widehat{S}(37)\big) = 13782649,\\
y_3 &= \breve{K}\big(\widehat{S}(19)\big) = 135137.
\end{aligned}$$

Then we have

$$y_1^2 + y_2^2 + y_3^2 = 3y_1y_2y_3.$$

Hence $\chi\big(\widehat{S}(5), \widehat{S}(37), \widehat{S}(19)\big)$ is a vertex in $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$.

Proof of Proposition 4.2.7. We have that

$$\begin{aligned}
S(3) &= S(0) \oplus S(2) = \widehat{S}\big(a^*(1)\big) \oplus \widehat{S}(2) = \widehat{S}(3), \text{ and}\\
S(4) &= S(2) \oplus S(1) = \widehat{S}(2) \oplus \widehat{S}\big(a(2)\big) = \widehat{S}(4).
\end{aligned}$$

This is a base of induction.

Assume that the proposition holds up to $n-1$-th level in the graph. Let $i \in \{1, \ldots, 2^{n-1}\}$. Let v be the vertex in the $n-1$-th level of the graph whose central element is $S(2^{n-1}{+}i)$. We have two cases, when i is even and when i is odd.

i even: Assume $i = 2k$ for some positive integer k. By the induction hypothesis we have that

$$v = \Big(\widehat{S}(2^{n-2}{+}k), S(2^{n-1}{+}i), \widehat{S}\big(a(2^{n-2}{+}k)\big)\Big).$$

We have that

$$\begin{aligned}
\mathcal{L}_\oplus(v) &= \Big(\widehat{S}(2^{n-2}{+}k), \widehat{S}(2^{n-2}{+}k) \oplus S(2^{n-1}{+}i), S(2^{n-1}{+}i)\Big),\\
\mathcal{R}_\oplus(v) &= \Big(S(2^{n-1}{+}i), S(2^{n-1}{+}i) \oplus \widehat{S}\big(a(2^{n-2}{+}k)\big), \widehat{S}\big(a(2^{n-2}{+}k)\big)\Big).
\end{aligned}$$

By the induction hypothesis and the ordering of the graph we have that

$$\begin{aligned}
S\big(2(2^{n-1}{+}i){-}1\big) &= \widehat{S}(2^{n-2}{+}k) \oplus S(2^{n-1}{+}i),\\
S\big(2(2^{n-1}{+}i)\big) &= S(2^{n-1}{+}i) \oplus \widehat{S}\big(a(2^{n-2}{+}k)\big).
\end{aligned}$$

We show that

$$\begin{aligned}
\widehat{S}\big(2(2^{n-1}{+}i){-}1\big) &= \widehat{S}(2^{n-2}{+}k) \oplus S(2^{n-1}{+}i),\\
\widehat{S}\big(2(2^{n-1}{+}i)\big) &= S(2^{n-1}{+}i) \oplus \widehat{S}\big(a(2^{n-2}{+}k)\big).
\end{aligned} \tag{4.3}$$

By definition of $\widehat{S}$ we have that

$$\begin{aligned}\widehat{S}\big(2(2^{n-1}+i)-1\big) &= \widehat{S}\big(a(2^{n-1}+i-1)\big) \oplus \widehat{S}(2^{n-1}+i)\\ &= \widehat{S}\Big(a\big(2(2^{n-2}+k)-1\big)\Big) \oplus \widehat{S}(2^{n-1}+i)\\ &= \widehat{S}(2^{n-2}+k) \oplus \widehat{S}(2^{n-1}+i)\\ &= \widehat{S}(2^{n-2}+k) \oplus S(2^{n-1}+i).\end{aligned}$$

The second and third equalities hold form the definition of a, and the final equality holds by the induction hypothesis. Similarly we have

$$\begin{aligned}\widehat{S}\big(2(2^{n-1}+i)\big) &= \widehat{S}(2^{n-1}+i) \oplus \widehat{S}\big(a(2^{n-1}+i)\big)\\ &= \widehat{S}\big(a(2^{n-1}+i)\big) \oplus \widehat{S}\Big(a\big(2(2^{n-2}+k)\big)\Big)\\ &= \widehat{S}\big(a(2^{n-1}+i)\big) \oplus \widehat{S}\big(a(2^{n-2}+k)\big)\\ &= S\big(a(2^{n-1}+i)\big) \oplus \widehat{S}\big(a(2^{n-2}+k)\big).\end{aligned}$$

Again, the second and third equalities hold form the definition of a, and the final equality holds by the induction hypothesis. Hence the equalities in Equation (4.3) hold.

i odd: A similar argument holds for the case when i is odd.

This concludes the proof. ■

Definition 4.2.9. The length of the sequence $S(n)$ is denoted $|S(n)|$.

Remark 4.2.10. We have from Proposition 4.2.7 that

$$\begin{aligned}|S(2n)| &= |S(n)| + \big|S\big(a(n)\big)\big|, \text{ and}\\ |S(2n-1)| &= |S\big(a(n-1)\big)| + |S(n)|,\end{aligned}$$

since $|S(0)| = |S(1)|$.

4.2.2 Symmetry of construction of sequences $\big(S(n)\big)$

We give some technical lemmas which significantly shorten the proof of Proposition 4.2.2. These lemmas use the related symmetry of the graph $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$ and Stern's diatomic sequence.

We start by noting a definition of Sterns diatomic sequence that shows its symmetry.

Remark 4.2.11. The table in Figure 4.4 is generated as follows:

Step 1. The first row contains two occurrences of the number 1

$$1 \qquad\qquad 1.$$

1								1
1				2				1
1		3		2		3		1
1	4	3	5	2	5	3	4	1

Figure 4.4: The first entries in Stern's diatomic sequence.

Step 2. The second row is obtained by taking a copy of the first row. Then each consecutive pair of numbers is added together, with the sum placed halfway between the two summands. Hence the second row is

$$1 \qquad 2 \qquad 1.$$

Step 3. The n-th row is started by taking a copy of the $n-1$-th row. Then each consecutive pair of numbers is added together, and the sum is placed halfway between the two summands. So for a consecutive pair in row $n-1$,

$$a \qquad\qquad\qquad b,$$

the corresponding elements in row n are

$$a \qquad a+b \qquad b.$$

Step 4. Stern's diatomic sequence is obtained from this table by omitting the right hand column of 1's, adding a leading 0 to the start, and reading the table left to right, top to bottom.

Note that the resulting table is symmetric about its central column.

We have a lemma relating on the length of a sequence $S(n)$.

Lemma 4.2.12. For $k \geq 2$ we have

$$|S(k)| = \left|S\big(a(k)\big)\right| + \left|S\big(a(k{-}1)\big)\right|.$$

Proof. We have

$$|S(2)| = 4 = 2|S(1)| = |S\big(a(2)\big)| + |S\big(a(1)\big)|.$$

This serves as a base of induction.

Assume $|S(k)| = |S\big(a(k)\big)|+|S\big(a(k{-}1)\big)|$ for all $k = 2, \ldots, N{-}1$, for some positive integer N. We have two cases:

N even: If $N = 2m$ for some positive integer m, then

$$\begin{aligned}|S(2m)| &= |S(m)| + \left|S\big(a(m)\big)\right| \\ &= \left|S\big(a(2m-1)\big)\right| + \left|S\big(a(2m)\big)\right| \\ &= \left|S\big(a(N-1)\big)\right| + \left|S\big(a(N)\big)\right|.\end{aligned}$$

The first equality holds by Proposition 4.2.7, and the following equalities hold by definition of a.

N odd: Let $N = 2m - 1$. Then

$$\begin{aligned}|S(2m-1)| &= \left|S\big(a^*(m-1)\big)\right| + |S(m)| \\ &= \left|S\big(a^*(2m-2)\big)\right| + \left|S\big(a(2m-1)\big)\right| \\ &= \left|S\big(a(N-1)\big)\right| + \left|S\big(a(N)\big)\right|.\end{aligned}$$

The first equality again holds by Proposition 4.2.7, and the following equalities hold by definition of a.

This concludes the proof. ■

Next we have a lemma relating the length $|S(n)|$ with Stern's diatomic sequence.

Lemma 4.2.13. For $k \geq 1$ we have

$$\left|S\big(a(k)\big)\right| = 2d_k.$$

Proof. We have that

$$\left|S\big(a(2)\big)\right| = \left|S\big(a(1)\big)\right| = 2 = 2d_1 = 2d_2.$$

This is a base of induction.

Assume $\left|S\big(a(k)\big)\right| = 2d_k$ for all $k = 1, \dots, N-1$, for some positive integer N. We have two cases:

N even: If $N = 2m$ for some positive integer m, then $d_{2m} = d_m$, and so we have $\left|S\big(a(2m)\big)\right| = \left|S\big(a(m)\big)\right|$. By the assumption we have $2d_m = \left|S\big(a(m)\big)\right|$, which happens if and only if

$$2d_N = \left|S\big(a(N)\big)\right|.$$

N odd: Let $N = 2m-1$ for some positive integer m. Then $\left|S\big(a(2m-1)\big)\right| = |S(m)|$. By Lemma 4.2.12 we have

$$|S(m)| = \left|S\big(a(m)\big)\right| + \left|S\big(a(m-1)\big)\right|.$$

By the assumption we have $\left|S\big(a(m)\big)\right| = 2d_m$ and $\left|S\big(a(m-1)\big)\right| = 2d_{m-1}$. Hence

$$\left|S\big(a(N)\big)\right| = |S(m)| = 2d_m + 2d_{m-1} = 2d_N.$$

This concludes the proof. ∎

The following lemma relates more specific sequences $S(n)$ with Stern's diatomic sequence.

Lemma 4.2.14. Let $n > 1$. For $i = 1, \dots, 2^{n-1}$ define the integers

$$k' = 6 \cdot 2^{n-2} + i - 1 \quad \text{and} \quad k'' = 6 \cdot 2^{n-2} - i + 1.$$

Then we have

$$\frac{|S(k'+1)|}{2} - d_{k'+1} = d_{k''}.$$

Proof. From Definition 4.2.3 we have that $k'+1 = a\big(2(k'+1)-1\big)$. Using Lemma 4.2.13 we then have

$$\begin{aligned} \frac{|S(k'+1)|}{2} = \frac{\left|S\Big(a\big(2(k'+1)-1\big)\Big)\right|}{2} &= d_{2(k'+1)-1} \\ &= d_{k'+1} + d_{k'}. \end{aligned}$$

It remains to show that

$$d_{k'} = d_{k''},$$

but this follows from the symmetry seen in Figure 4.4, see Remark 4.2.11. ∎

We give an example of Lemmas 4.2.14, 4.2.12 and 4.2.13.

Example 58. We give an example for $S(15) = (1,1,2,2,2,2,1,1,2,2,2,2,2,2)$.

(*Lemma* 4.2.12) We have $a(15) = 8$, $a(14) = 4$, $S(8) = (1,1,2,2,2,2,2,2)$, $S(4) = (1,1,2,2,2,2)$, and hence

$$\begin{aligned} 14 &= |S(15)|, \\ 8 + 6 &= |S(8)| + |S(4)|. \end{aligned}$$

(*Lemma* 4.2.13) We have that $d_{15} = 4$ and $d_{14} = 3$, and so

$$\begin{aligned} 8 &= |S\big(a(15)\big)| = |S(8)| = 2d_{15} = 8, \\ 6 &= |S\big(a(14)\big)| = |S(4)| = 2d_4 = 6. \end{aligned}$$

(*Lemma* 4.2.14) Now $k' = 14 = 6 \cdot 2^{3-2} + 3 - 1$, so in the notation of Lemma 4.2.14 we have that $n = 3$ and $i = 3$. Furthermore we have that $k'' = 6 \cdot 2^{3-2} - 3 + 1 = 10$ and $d_{10} = 3$. Hence

$$\begin{aligned} \frac{S(14+1)}{2} - d_{15} &= 7 - 4 \\ &= 3 = d_{10}. \end{aligned}$$

Let $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_m)$. Note that

$$\overline{\alpha\beta} = (b_m \ldots, b_1, a_n, \ldots, a_1) = \overline{\beta} \oplus \overline{\alpha}.$$

Let Z be the function on triples of sequences $(\alpha, \alpha\beta, \beta)$ defined by

$$Z(\alpha, \alpha\beta, \beta) = \alpha\beta.$$

The next definition and proposition investigate the symmetry of sequences in a Markov graph.

Definition 4.2.15. Let $v = (\mu, \mu\nu, \nu)$, and consider a vertex

$$w_1 = \mathcal{L}_\oplus^{\delta_{2n}} \mathcal{R}_\oplus^{\delta_{2n-1}} \ldots \mathcal{L}_\oplus^{\delta_2} \mathcal{R}_\oplus^{\delta_1}(v)$$

in the graph $\mathcal{G}_\oplus(\mu, \nu)$, where $\delta_1, \ldots, \delta_{2n-1}$ are positive integers , and n, δ_1, and δ_{2n} are non-negative integers. Recall that the sequence $\mathcal{L}_\oplus^{\delta_{2n}} \mathcal{R}_\oplus^{\delta_{2n-1}} \ldots \mathcal{L}_\oplus^{\delta_2} \mathcal{R}_\oplus^{\delta_1}$ is called the Farey sequence of w_1. We call $\mathcal{R}_\oplus^{\delta_{2n}} \mathcal{L}_\oplus^{\delta_{2n-1}} \ldots \mathcal{R}_\oplus^{\delta_2} \mathcal{L}_\oplus^{\delta_1}$ the *reverse Farey sequence of* w_1.

Proposition 4.2.16. *Let v_1 be the vertex $(\mu, \mu\nu, \nu)$ in a Markov graph $\mathcal{G}_\oplus(\mu, \nu)$. Let v_2 be the vertex $(\overline{\nu}, \overline{\nu} \oplus \overline{\mu}, \overline{\mu})$ in the graph $\mathcal{G}_\oplus(\overline{\nu}, \overline{\mu})$. Let*

$$\begin{aligned} w_1 &= \mathcal{L}_\oplus^{\delta_{2n}} \mathcal{R}_\oplus^{\delta_{2n-1}} \ldots \mathcal{L}_\oplus^{\delta_2} \mathcal{R}_\oplus^{\delta_1}(v_1), \\ w_2 &= \mathcal{R}_\oplus^{\delta_{2n}} \mathcal{L}_\oplus^{\delta_{2n-1}} \ldots \mathcal{R}_\oplus^{\delta_2} \mathcal{L}_\oplus^{\delta_1}(v_2). \end{aligned}$$

Then

$$Z(w_1) = \overline{Z(w_2)}.$$

Proof. We use induction. We see that

$$\begin{aligned} Z\big(\mathcal{L}_\oplus(v_1)\big) &= \mu\mu\nu = \overline{\overline{\nu} \oplus \overline{\mu} \oplus \overline{\mu}} = \overline{Z\big(\mathcal{R}_\oplus(v_2)\big)}, \\ Z\big(\mathcal{R}_\oplus(v_1)\big) &= \mu\nu\nu = \overline{\overline{\nu} \oplus \overline{\nu} \oplus \overline{\mu}} = \overline{Z\big(\mathcal{L}_\oplus(v_2)\big)}. \end{aligned}$$

This is the base of our induction.

Assume for some positive integers n and $\delta_1, \ldots, \delta_{2n-1}$, and non-negative integers δ_1 and δ_{2n} we have that

$$Z\Big(\mathcal{L}_\oplus^{\delta_{2n}} \mathcal{R}_\oplus^{\delta_{2n-1}} \ldots \mathcal{L}_\oplus^{\delta_2} \mathcal{R}_\oplus^{\delta_1}(v_1)\Big) = \overline{Z\Big(\mathcal{R}_\oplus^{\delta_{2n}} \mathcal{L}_\oplus^{\delta_{2n-1}} \ldots \mathcal{R}_\oplus^{\delta_2} \mathcal{L}_\oplus^{\delta_1}(v_2)\Big)}$$

Let $w_1 = \mathcal{L}_\oplus^{\delta_{2n}} \mathcal{R}_\oplus^{\delta_{2n-1}} \ldots \mathcal{L}_\oplus^{\delta_2} \mathcal{R}_\oplus^{\delta_1}(v_1)$ and $w_2 = \mathcal{R}_\oplus^{\delta_{2n}} \mathcal{L}_\oplus^{\delta_{2n-1}} \ldots \mathcal{R}_\oplus^{\delta_2} \mathcal{L}_\oplus^{\delta_1}(v_2)$. We have for some sequences α and β that

$$w_1 = (\alpha, \alpha\beta, \beta) \qquad \text{and} \qquad w_2 = (\overline{\beta}, \overline{\beta} \oplus \overline{\alpha}, \overline{\alpha}),$$

with w_2 being defined by the induction hypothesis. Then

$$\begin{aligned} Z\big(\mathcal{L}_\oplus(w_1)\big) &= \alpha\alpha\beta = \overline{\overline{\beta} \oplus \overline{\alpha} \oplus \overline{\alpha}} = Z\big(\mathcal{R}_\oplus(w_2)\big), \\ Z\big(\mathcal{R}_\oplus(w_1)\big) &= \alpha\beta\beta = \overline{\overline{\beta} \oplus \overline{\beta} \oplus \overline{\alpha}} = Z\big(\mathcal{L}_\oplus(w_2)\big). \end{aligned}$$

This completes the induction, and the proof. ■

Remark 4.2.17. Let $v = (\mu, \mu\nu, \nu)$. The vertices in the n-th level of the ordered Markov graph $\mathcal{G}_{\oplus}(\mu, \nu)$ are given by

$$\mathcal{L}_{\oplus}^{\delta_{2k}}\mathcal{R}_{\oplus}^{\delta_{2k-1}} \dots \mathcal{L}_{\oplus}^{\delta_2}\mathcal{R}_{\oplus}^{\delta_1}(v),$$

where

$$\sum_{i=0}^{2k} \delta_i = n - 1.$$

Note that the sequence $S(2^n{+}1)$ is $Z\big(\mathcal{L}_{\oplus}^{n-1}(v)\big)$ and $S(2^{n+1}) = Z\big(\mathcal{R}_{\oplus}^{n-1}(v)\big)$.

Remark 4.2.18. Note that in the n-th level of the graph we have sequences $S(2^n + i)$ for $i \in \{1, \dots, 2^n\}$. The central two elements in the sequence

$$\Big(S(2^n + i)\Big)_{i=1}^{2^n}$$

are $S(2^n{+}2^{n-1})$ and $S(2^n{+}2^{n-1}{+}1)$. These are given by

$$\begin{aligned} S(2^n{+}2^{n-1}) &= S\big(2(2^{n-1}){+}2^{n-1}\big) \\ &= S\big(3 \cdot 2^{n-1}\big) \\ &= S\big(6 \cdot 2^{n-2}\big) \end{aligned}$$

and $S(2^n{+}2^{n-1}) = S(6 \cdot 2^{n-2}{+}1)$. Hence, if $k = 6 \cdot 2^{n-2}{+}i$ and $k' = 6 \cdot 2^{n-2}{-}i{+}1$, then we have that $S(k)$ is the reverse sequence of $S(k')$, by Lemma 4.2.14

Example 59. Begin with $S(15) = (1, 1, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2, 2)$. We have $k = 15 = 6 \cdot 2^{3-2} + 3$ and $m = 6 \cdot 2^{3-2} - 3 + 1 = 10$. Then

$$\begin{aligned} S(10) &= (1, 1, 1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 2, 2), \\ \overline{S(10)} &= (2, 2, 1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 1, 1), \\ S(15) &= (1, 1, 2, 2, 2, 2, 1, 1, 2, 2, 2, 2, 2, 2), \end{aligned}$$

and so

$$S_{(1,1),(2,2)}(15) = \overline{S_{(2,2),(1,1)}(10)}.$$

4.2.3 Canonical form for Markov sequences

In the previous subsection we saw how to define the integers a and b from a positive integer n such that

$$\big(S(a), S(n), S(b)\big)$$

is a Markov triple. We take this idea further in this subsection.

We split each sequence into two parts; one part contains a single Markov sequence (called the *base element*), and the second part contains a separate Markov sequence repeated at least twice (called the *repeating element*).

Consider such a Markov sequence $S(n) = \alpha \oplus \beta^i$, where α is the base element and β is the repeating element (both Markov sequences with $i > 1$). The value of this representation, given in Lemmas 4.2.20 and 4.2.22, is that when we concatenate a Markov sequence with its repeating element the following important facts hold:

1. $\alpha \oplus \beta^{i+1}$ is a Markov sequence.

2. If $C_{d_n}\big(\alpha \oplus \beta^i\big)$ is palindromic, then so is $C_{d_n}\big(\alpha \oplus \beta^{i+1}\big)$.

One could also have a case where a Markov sequence is split $S(m) = \alpha^j \oplus \beta$. This is dealt with largely through Remark 4.2.18. Note that $S\big(a(n)\big)^i$ is the sequence $S\big(a(n)\big)$ repeated i times

We start by defining some decreasing sequences of positive integers.

Definition 4.2.19. (*i*) Let n be an even integer with $n > 2$. Let $\big(t_i\big)_{i=1}^{\infty}$ be the sequence with $t_1 = n$ and

$$t_i = \frac{t_{i-1}}{2},$$

for $i > 1$. We define N to be the number such that t_{N-1} is a positive integer, and t_N is odd. Let $k_i = t_i$ for $i = 1, \ldots, N$, and let $k_{N+1} = (k_N + 1)/2$. If $k_N = 1$ then we call the sequence

$$(k_1 \ldots, k_N)$$

the *sequence induced by* n. If $k_N \neq 1$ then we call the sequence

$$(k_1 \ldots, k_{N+1})$$

the *sequence induced by* n.

(*ii*) Let n be an odd integer with $n > 2$. Let $\big(t_i\big)_{i=1}^{\infty}$ be the sequence with $t_1 = n$ and

$$t_i = \frac{t_{i-1} + 1}{2},$$

for $i > 1$. We define N to be the number such that t_{N-1} is a positive integer and t_N is even. Let $k_i = t_i$ for $i = 1, \ldots, N$ and let $k_{N+1} = (k_N)/2$. We call the sequence

$$(k_1 \ldots, k_{N+1})$$

the *sequence induced by* n.

Example 60. Let $n_1 = 13$. The sequence induced by n_1 is

$$(13, 7, 4, 2).$$

Let $n_2 = 16$. The sequence induced by n_2 is

$$(16, 8, 4, 2, 1).$$

We have lemmas giving us a splitting of a Markov sequence $S(n)$ dependent on the sequence induced by n.

Lemma 4.2.20. Let $k = k_1 > 2$ be an even positive integer, and let i be a positive integer such that the numbers

$$k_j = \frac{k_{j-1}}{2}$$

are positive integers for $j = 1, \ldots, i$, and k_i is odd. Let $k_{i+1} = (k_i{+}1)/2$. Then

$$S(k) = \begin{cases} S(0) \oplus S(1)^i, & \text{if } k_{i+1} = 1, \\ S\big(a^*(k_{i+1}{-}1)\big) \oplus S(k_{i+1})^i, & \text{if } k_{i+1} > 1, \end{cases}$$

where the power i indicates a sequence concatenated i times.

Definition 4.2.21. For even k we call the splitting of the Markov sequence $S(k)$ in Lemma 4.2.20 the *canonical splitting of* $S(k)$.

Example 61. For $k_1 = 10$ we have $S(10) = (1,1,1,1,1,1,2,2,1,1,1,1,2,2)$, and the sequence induced by k_1 is $(k_i)_1^3 = (10,5,3)$. Note that $a^*(k_3{-}1) = 0$, so $S\big(a^*(k_3{-}1)\big) = S(0) = (1,1)$. Also we have $S(3) = (1,1,1,1,2,2)$, and hence

$$S(10) = S(0) \oplus S(3)^2 = (1,1) \oplus (1,1,1,1,2,2)^2.$$

Note also that

$$S(20) = S(2 \cdot 10) = S(0) \oplus S(3)^2 \oplus S(3) = (1,1) \oplus (1,1,1,1,2,2)^3.$$

Finally, we see that

$$\begin{aligned} C_3\big(S(10)\big) &= (1,1,1,2,2,1,1,1,1,2,2,1,1,1), \\ C_3\big(S(20)\big) &= (1,1,1,2,2,1,1,1,1,2,2,1,1,1,1,2,2,1,1,1). \end{aligned}$$

Proof of Lemma 4.2.20. Through repeated application of Proposition 4.2.7 we have that

$$S(k) = S(k_2) \oplus S\big(a(k_2)\big) = S(k_i) \oplus S\big(a(k_i)\big)^{i-1}.$$

If $k_{i+1} = 1$ then $k_i = 2$, and so

$$S(k) = S(2) \oplus S\big(a(2)\big)^{i-1} = S(0) \oplus S(1)^i.$$

If $k_{i+1} > 1$, then since $k_i = 2k_{i+1}{-}1$, we have that

$$S(k_i) = S\big(a^*(k_{i+1}{-}1)\big) \oplus S(k_{i+1}),$$

and so

$$S(k) = S\big(a^*(k_{i+1}{-}1)\big) \oplus S(k_{i+1})^i,$$

which proves the lemma. ∎

Lemma 4.2.22. Let $k = k_1 > 1$ be an odd positive integer, and let i be a positive integer such that the numbers

$$k_j = \frac{k_{j-1}+1}{2}$$

for $j = 1, \dots, i$ are positive integers, and $k_i = (k_{i-1}+1)/2$ is even. Let $k_{i+1} = k_i/2$. Then

$$S(k) = \begin{cases} S(k_{i+1})^i \oplus S\big(a(k_{i+1})\big), & \text{if } k_i > 2,\\ S(0)^i \oplus S(1), & \text{if } k_i = 2.\end{cases}$$

Definition 4.2.23. For odd k we call the splitting of the Markov sequence $S(k)$ in Lemma 4.2.22 the *canonical splitting of* $S(k)$.

Example 62. For $k_1 = 13$ we have $S(13) = (1,1,2,2,1,1,2,2,1,1,2,2,2,2)$, and the sequence induced by k_1 is $(k_i)_1^4 = (13,7,4,2)$. Note that $S(2) = (1,1,2,2)$ and $S\big(a(2)\big) = S(1)$, and so

$$S(13) = S(2)^3 \oplus S(0) = (1,1,2,2)^3 \oplus (2,2).$$

Note also that

$$S(25) = S(2\cdot 13 - 1) = S(2)^4 \oplus S(2) = (1,1,2,2)^4 \oplus (2,2).$$

Finally we have

$$\begin{aligned} C_5\big(S(13)\big) &= (1,2,2,1,1,2,2,2,2,1,1,2,2,1),\\ C_7\big(S(25)\big) &= (2,1,1,2,2,1,1,2,2,2,2,1,1,2,2,1,1,2). \end{aligned}$$

Proof of Lemma 4.2.22. Applying the result of Proposition 4.2.7 we have

$$\begin{aligned} S(k) &= S\big(a^*(k_2-1)\big) \oplus S(k_2)\\ &= S\big(a^*(2k_3-2)\big) \oplus S\big(a^*(k_3-1)\big) \oplus S(k_3)\\ &= S\big(a^*(k_i-1)\big)^{i-1} \oplus S(k_i). \end{aligned}$$

If $k_i > 2$ then

$$\begin{aligned} S(k) &= S\big(a^*(2k_{i+1}-1)\big)^{i-1} \oplus S(2k_{i+1})\\ &= S(k_{i+1})^i \oplus S\big(a(k_{i+1})\big). \end{aligned}$$

If $k_i = 2$ then

$$\begin{aligned} S(k) &= S\big(a^*(k_i-1)\big)^{i-1} \oplus S(k_i)\\ &= S(0)^{i-1} \oplus S(2)\\ &= S(0)^i \oplus S(1). \end{aligned}$$

This completes the proof. ■

Remark 4.2.24. Let $k > 2$. Note that if we have a splitting for a Markov sequence $S(k)$

$$S(k) = S(a) \oplus S(b)$$

for some positive integers a and b, then this cannot be a canonical splitting. This follows since each integer $k > 2$ induces a sequence $(k_i)_{i=1}^{n}$, where n must be greater than 1. The canonical splitting then includes some sequence to the power n.

The final lemma before the proof of Proposition 4.2.2 is purely technical, and it shortens the notation used in the proof.

Lemma 4.2.25. Assume that the d_n-th circular shift of $S(n)$ is palindromic for all $n = 1, \ldots, k_1-1$, for some positive even integer k_1. Let $(k_1, \ldots, k_{i+1})$ be the sequence induced by k. Assume also that $k_{i+1} > 1$. Let $L = d_{k_2}$, and let

$$R = L + \frac{\left|S\big(a(k_{i+1}-1)\big)\right| + (i-1)|S(k_{i+1})|}{2}.$$

Then $R > \left|S\big(a(k_{i+1}-1)\big)\right|$.

Proof. If $i > 2$ we have that

$$L = d_{k_2} = d_{2k_3} = d_{k_3} = \ldots = d_{k_i} = d_{2k_{i+1}-1} = d_{k_{i+1}} + d_{k_{i+1}-1}.$$

Also

$$\left|S\big(a(k_{i+1}-1)\big)\right| = 2d_{k_{i+1}-1}.$$

Hence

$$R = d_{k_{i+1}} + d_{k_{i+1}-1} + \frac{2d_{k_{i+1}-1} + (i-1)|S(k_{i+1})|}{2} > 2d_{k_{i+1}-1} = \left|S\big(a(k_{i+1}-1)\big)\right|.$$

If $i = 2$ then

$$L = d_{k_2} = d_{2k_3-1} = d_{k_3} + d_{k_3-1}.$$

Hence

$$R = d_{k_3} + d_{k_3-1} + d_{k_3-1} + \frac{|S(k_3)|}{2} > 2d_{k_3-1} = \left|S\big(a(k_3 - 1)\big)\right|.$$

■

4.2.4 Proof of Proposition 4.2.2

Now we prove Proposition 4.2.2.

Proof of Proposition 4.2.2. We must show that $C_{d_n}\big(S(n)\big)$ is palindromic. To do this we use induction on n.

For $n \in \{0, 1, 2\}$ we have

$$\begin{aligned} C_{d_0}(1,1) &= C_0(1,1) = (1,1), \\ C_{d_1}(2,2) &= C_1(2,2) = (2,2), \\ C_{d_2}(1,1,2,2) &= C_1(1,1,2,2) = (1,2,2,1). \end{aligned}$$

This serves as a base of induction.

Assume the statement holds for all $n = 3, \ldots, k-1$, for some integer $k > 3$. We have two cases, for when k is even or odd. In either case, we denote the elements of the sequence $S(k)$ by λ's, so

$$S(k) = (\lambda_1, \ldots, \lambda_{|S(k)|}).$$

(i) Let $k = k_1$ be even. Let $(k_1, \ldots, k_{i+1})$ be the sequence induced by k. If $k_{i+1} = 1$ then $k_i = 2$, and so $k_1 = 2^i$. Hence $d_{k_1} = d_1 = 1$. We have that

$$S(k) = S(0) \oplus S(1)^i,$$

and so

$$C_1\big(S(k)\big) = (1, 2, 2, \ldots, 2, 2, 1),$$

which is palindromic.

If $k_{i+1} > 1$ then let N and M be the lengths of the sequences $S\big(a^*(k_{i+1}-1)\big)$ and $S(k_{i+1})$ respectively, so

$$N = \big|S\big(a^*(k_{i+1}-1)\big)\big| \qquad \text{and} \qquad M = |S(k_{i+1})|.$$

We have that

$$\begin{aligned} S(k_1) &= S\big(a^*(k_{i+1}-1)\big) \oplus S(k_{i+1})^i, \\ S(k_2) &= S\big(a^*(k_{i+1}-1)\big) \oplus S(k_{i+1})^{i-1}. \end{aligned}$$

By the induction hypothesis we have that

$$C_{d_{k_2}}\big(S(k_2)\big) = C_{d_{k_2}}\Big(S\big(a^*(k_{i+1}-1)\big) \oplus S(k_{i+1})^{i-1}\Big)$$

is palindromic. The inductive step is to show that

$$C_{d_{k_1}}\big(S(k_1)\big) = C_{d_{k_1}}\Big(S\big(a^*(k_{i+1}-1)\big) \oplus S(k_{i+1})^{i}\Big)$$

is also palindromic.

Let $L = d_{k_2}$, and set

$$R = L + \frac{N + (i-1)M}{2}.$$

This is L added to half the length of the sequence $S(k_2)$. We already have from the induction hypothesis the following relations for the elements of $S(k_2)$

$$\begin{gathered} \lambda_L = \lambda_{L+1}, \quad \lambda_{L-1} = \lambda_{L+2}, \ldots, \quad \lambda_1 = \lambda_{2L}, \\ \lambda_{|S(k_2)|} = \lambda_{2L+1}, \quad \lambda_{|S(k_2)|-1} = \lambda_{2L+2}, \ldots, \quad \lambda_{R+2} = \lambda_{R-1}, \quad \lambda_{R+1} = \lambda_R. \end{gathered}$$

Since $S(k_1)$ is obtained by the concatenation

$$\Big(S\big(a^*(k_{i+1}-1)\big) \oplus S(k_{i+1})^{i-1}\Big) \oplus S(k_{i+1}),$$

we immediately derive the following relations for elements of $S(k_1)$

$$\lambda_L = \lambda_{L+1}, \quad \lambda_{L-1} = \lambda_{L+2}, \ldots, \quad \lambda_1 = \lambda_{2L},$$
$$\lambda_{|S(k_1)|} = \lambda_{2L+1}, \quad \lambda_{|S(k_1)|-1} = \lambda_{2L+2}, \ldots, \quad \lambda_{R+M+2} = \lambda_{R-1}, \quad \lambda_{R+M+1} = \lambda_R.$$

The subsequence of $S(k_1)$

$$(\lambda_{R+1}, \lambda_{R+2}, \ldots, \lambda_{R+M-1}, \lambda_{R+M})$$

is simply a circular shift of $S(k_{i+1})$.

We must show the following condition

$$\lambda_R = \lambda_{R+M+1}, \ldots, \lambda_{R+M/2} = \lambda_{R+M/2+1}. \tag{4.4}$$

To do this we note that, with $R>N$ from Lemma 4.2.25, we can ignore the sequence $S\big(a^*(k_{i+1}-1)\big)$ at the start of $S(k)$ (by subtracting its length N from R), remove any excess copies of the sequence $S(k_{i+1})$ (by working modulo M), and get that the condition in Equation (4.4) is equivalent to having

$$R+\frac{M}{2}-N \mod M = \begin{cases} d_{k_{i+1}}, & \text{if } i \text{ is even,} \\ d_{k_{i+1}} + \frac{|S(k_{i+1})|}{2}, & \text{if } i \text{ is odd.} \end{cases} \tag{4.5}$$

That is to say, if i is even we show that

$$C_{d_{k_{i+1}}}\big(S(k_{i+1})\big) = (\lambda_R, \ldots, \lambda_{R+M/2}, \lambda_{R+M/2+1}, \ldots, \lambda_{R+M+1}),$$

and if i is odd we show that

$$C_{d_{k_{i+1}}+|S(k_{i+1})|/2}\big(S(k_{i+1})\big) = (\lambda_R, \ldots, \lambda_{R+M/2}, \lambda_{R+M/2+1}, \ldots, \lambda_{R+M+1}),$$

and since both of these sequences are palindromic by the induction hypothesis, we have the condition of Equation (4.4) holds.

Substituting in for R, M, and N, the left hand side of Equation (4.5) becomes

$$d_{k_2}+\frac{|S(k_2)|}{2}+\frac{|S(k_{i+1})|}{2}-\big|S\big(a^*(k_{i+1}-1)\big)\big| \mod |S(k_{i+1})|.$$

Recall that $d_{k_2} = d_{k_3} = \ldots = d_{k_i} = d_{2k_{i+1}-1} = d_{k_{i+1}}+d_{k_{i+1}-1}$ and

$$|S(k_2)| = \big|S\big(a^*(k_{i+1}-1)\big)\big| + (i-1)|S(k_{i+1})|,$$

and so

$$\begin{aligned} &R+\frac{M}{2}-N \mod M = \\ &d_{k_{i+1}} + d_{k_{i+1}-1} + \frac{i|S(k_{i+1})|}{2} - \frac{\big|S\big(a^*(k_{i+1}-1)\big)\big|}{2} \mod |S(k_{i+1})|. \end{aligned}$$

If i is even then this becomes

$$d_{k_{i+1}} + d_{k_{i+1}-1} - \frac{\big|S\big(a^*(k_{i+1}-1)\big)\big|}{2} \equiv d_{k_{i+1}} \mod |S(k_{i+1})|,$$

which is true by Lemma 4.2.13.

If i is odd, then we have

$$d_{k_{i+1}} + d_{k_{i+1}-1} + \frac{|S(k_{i+1})|}{2} - \frac{\left|S\big(a^*(k_{i+1}-1)\big)\right|}{2} \equiv$$
$$d_{k_{i+1}} + \frac{|S(k_{i+1})|}{2} \mod |S(k_{i+1})|,$$

which is again true by Lemma 4.2.13. So we have that the d_k-th circular shift of $S(k)$ is palindromic, and the induction holds.

(ii) The proof for the case when $k = k_1$ is odd is equivalent to the even case by Lemma 4.2.14, and Remark 4.2.18. This concludes the proof of Proposition 4.2.2. ■

4.2.5 Proof of Theorem 4.1.27

In this subsection we finalise the proof of Theorem 4.1.27. We start with the following lemma, coming from [23].

Remark 4.2.26. Recall the definition of the half sequences $\lfloor\Lambda\rfloor$ and $\lceil\Lambda\rceil$. Clearly, $\overline{\lfloor\Lambda\rfloor} = \lceil\Lambda\rceil$ if Λ is palindromic.

Proof of Theorem 4.1.27. Let $n > 2$ be a positive integer. Let $\Lambda_i \in \{A, B\}$ such that
$$S_n(A, B) = \Lambda_1 \dots \Lambda_N$$
for some positive integer N. Let $A = (a, a)$ and $B = (b, b)$. Let $\lambda_i \in \{a, b\}$ such that
$$S_n(A, B) = \lambda_1, \dots, \lambda_{2N}.$$
Note that $\Lambda_k = (\lambda_{2k-1}, \lambda_{2k})$. We have two cases, for when d_n is odd or even.

Let d_n be odd. Let $k < N$ be a positive integer such that

$$C_{d_n}(S_n) = \lambda_{d_n+1}\Lambda_k \dots \Lambda_N\Lambda_1 \dots \Lambda_{k-2}\lambda_{d_n},$$

which is palindromic by Proposition 4.2.2. Then

$$\begin{aligned} 2k - 1 &= d_n + 2, \\ k &= \frac{d_n + 1}{2} + 1, \\ k &= \left\lceil \frac{d_n}{2} \right\rceil + 1. \end{aligned}$$

Hence
$$C_{d_n}(S_n) = \lambda_{d_n+1}\Lambda_{1+\lceil d_n/2\rceil} \dots \Lambda_N\Lambda_1 \dots \Lambda_{\lceil d_n/2\rceil-1}\lambda_{d_n}.$$

Let us relabel $\Lambda_{1+\lceil d_n/2\rceil}$ to Γ_1 such that

$$C_{d_n}(S_n) = \lambda_{d_n+1}\Gamma_1 \dots \Gamma_{N-1}\lambda_{d_n}.$$

Then

$$\Gamma_1 = \Gamma_{N-1}, \ldots, \Gamma_{\lfloor (N-1)/2 \rfloor} = \Gamma_{\lceil (N-1)/2 \rceil + 1}.$$

Substituting A and B with any palindromic sequences μ and ν does not change these equalities, neither does letting $\lambda_{d_n} = \lceil \Lambda \rceil$, $\lambda_{d_n} = \lfloor \Lambda \rfloor$ for any palindromic Λ. Hence

$$\lceil \Lambda_{\lceil d_n/2 \rceil} \rceil \Lambda_{\lceil d_n/2 \rceil + 1} \ldots \Lambda_N \Lambda_1 \ldots \Lambda_{\lceil d_n/2 \rceil - 1} \lfloor \Lambda_{\lceil d_n/2 \rceil} \rfloor$$

is palindromic.

Now let d_n be even. Let $k < N$ be a positive integer such that

$$\begin{aligned} C_{d_n}(S_n) &= \Lambda_k \ldots \Lambda_N \Lambda_1 \ldots \Lambda_{k-1}, \\ &= \lambda_{d_n+1} \lambda_{d_n+2} \ldots \lambda_{2N} \lambda_1 \ldots \lambda_{d_n}. \end{aligned}$$

Then $k = 1 + d_n/2$, and

$$C_{d_n}(S_n) = \Lambda_{1+d_n/2} \ldots \Lambda_N \Lambda_1 \ldots \Lambda_{d_n/2}$$

is palindromic by Proposition 4.2.2. Let us relabel $\Lambda_{1+d_n/2}$ to Γ_1 so that

$$C_{d_n}(S_n) = \Gamma_1 \ldots \Gamma_N.$$

Then

$$\Gamma_1 = \Gamma_N, \ldots, \Gamma_{\lfloor N/2 \rfloor} = \Gamma_{\lceil N/2 \rceil + 1}.$$

Replacing A and B with any palindromic sequences μ and ν does not change these equalities, and so

$$\Lambda_{d_n/2+1} \Lambda_{d_n/2+2} \ldots \Lambda_N \Lambda_1 \ldots \Lambda_{d_n/2}$$

is palindromic. This completes the proof.

■

4.3 Markov graph relations

In this section we use Markov graphs to generate graphs of forms, matrices, and general Markov numbers. The relations between Markov graphs, graphs of forms, and graphs of matrices are the same as in Chapter 3, as seen in Theorems 3.3.7 and 3.3.14.

In Subsection 4.3.1 we note the Markov value of previously defined general reduced Markov forms.

We define graphs of general reduced Markov matrices in Subsection 4.3.2. We present a theorem constructing all reduced matrices in the integer conjugacy classes of matrices defined by periods of LLS sequences.

We define the graph of general Markov numbers in Subsection 4.3.3. We show that there is no simple recurrence relation to generate graphs of general Markov numbers directly from vertices of integers. Finally we relate the graphs of general reduced Markov matrices and forms to the graph of general Markov numbers.

Let μ and ν be even length sequences such that $\mathcal{G}_{\oplus}(\mu,\nu)$ is a Markov graph. Recall from Chapter 3 that

$$\begin{aligned}\Gamma\big(\mathcal{G}_{\oplus}(\mu,\nu)\big) &= \mathcal{G}_{\bullet}\big(M_\mu, M_\nu\big),\\ \Omega\big(\mathcal{G}_{\bullet}\big(M_\mu, M_\nu\big)\big) &= \mathcal{G}_{\theta}\big(f_\mu, f_\nu\big),\\ \Phi\big(\mathcal{G}_{\oplus}(\mu,\nu)\big) &= \mathcal{G}_{\theta}\big(f_\mu, f_\nu\big).\end{aligned}$$

4.3.1 General forms

Recall the definition for a graph of general reduced Markov forms $\mathcal{G}_{\theta}\big(f_\mu, f_\nu\big)$ from Section 4.1. We note the Markov value of these forms, and give an example.

Remark 4.3.1. From Theorem 4.1.14 we have that every form f_α in every vertex of a graph of general reduced Markov forms has

$$m\big(f_\alpha\big) = f_\alpha(1,0) = \breve{K}(\alpha).$$

If $\alpha = (a_1,\ldots,a_n)$ then the Markov value of f_α is

$$\frac{\sqrt{\big(K_1^n(\alpha)+K_2^{n-1}(\alpha)\big)^2-4}}{K_1^{n-1}(\alpha)}.$$

Example 63. Let $\mu = (1,1)$ and $\nu = (3,2,1,1,2,3)$ be sequences. Then the graph $\mathcal{G}_{\oplus}(\mu,\nu)$ is Markov. Let $\alpha = \mu\mu\nu$. Then

$$f_\alpha(x,y) = 259x^2 + 719xy - 535y^2.$$

The element on the Markov spectrum associated with this form is

$$\frac{\sqrt{1071221}}{259}.$$

4.3.2 Matrices

In this subsection we define the graph of general reduced Markov matrices from a Markov graph. We have the following definition.

Definition 4.3.2. Let $\mathcal{G}_{\oplus}(\mu,\nu)$ be a Markov graph. For a sequence $\alpha = (a_i)_1^n$ in a vertex of $\mathcal{G}_{\oplus}(\mu,\nu)$, recall the definition of M_α, the matrix associated with α,

$$M_\alpha = \begin{pmatrix} K_2^{n-1}(\alpha) & K_2^n(\alpha) \\ K_1^{n-1}(\alpha) & K_1^n(\alpha) \end{pmatrix}.$$

The graph of matrices $\Gamma\big(\mathcal{G}_{\oplus}(\mu,\nu)\big) = \mathcal{G}_{\bullet}\big(M_\mu, M_\nu\big)$ is called the *graph of general reduced Markov matrices.* We have that these graphs are equal by Theorem 3.3.14 and the proof of Theorem 3.2.20. We call any matrix M in any vertex of the graph $\mathcal{G}_{\bullet}\big(M_\mu, M_\nu\big)$ a *general reduced Markov matrix.*

By this definition, $\mathcal{G}_{\bullet}\big(M_{(1,1)}, M_{(2,2)}\big)$ is also a graph of general reduced Markov matrices.

As an aside we construct all reduced matrices for periodic LLS sequences.

Definition 4.3.3. Let $\alpha = (a_1, \dots, a_n)$ be the period of an LLS sequence. Let $\mathcal{P}(\alpha)$ be the set of all matrices $\mathrm{SL}(2,\mathbb{Z})$ conjugate to

$$\begin{pmatrix} K_2^{n-1}(\alpha) & K_2^n(\alpha) \\ K_1^{n-1}(\alpha) & K_1^n(\alpha) \end{pmatrix}.$$

Let $\mathcal{Q}(\alpha)$ be the set of all matrices $\mathrm{SL}(2,\mathbb{Z})$ conjugate to

$$\begin{pmatrix} K_2^{n-1}(\alpha) & K_1^{n-1}(\alpha) \\ K_2^n(\alpha) & K_1^n(\alpha) \end{pmatrix}.$$

From [52, Proposition 7.11,Corollary 7.17], we have that two $\mathrm{SL}(2,\mathbb{Z})$ matrices with positive eigenvalues are $\mathrm{SL}(2,\mathbb{Z})$ similar if their LLS sequences coincide.

Theorem 4.3.4. *Let $\alpha = (a_1, \dots, a_n)$ be the period of an LLS sequence. Then every reduced matrix in $\mathcal{P}(\alpha)$ is given by*

$$\begin{pmatrix} K_2^{n-1}(\alpha_i) & K_2^n(\alpha_i) \\ K_1^{n-1}(\alpha_i) & K_1^n(\alpha_i) \end{pmatrix},$$

and the reduced matrices in $\mathcal{Q}(\alpha)$ are given by

$$\begin{pmatrix} K_2^{n-1}(\alpha_i) & K_1^{n-1}(\alpha_i) \\ K_2^n(\alpha_i) & K_1^n(\alpha_i) \end{pmatrix},$$

where α_i is the i-th circular shift of α for $i = 1, \dots, n$. Also, α is evenly palindromic if and only if $\mathcal{P} = \mathcal{Q}$.

Example 64. Let $\alpha = (1, 2, 3, 4)$. Then the reduced matrices of $\mathcal{P}(\alpha)$ are

$$\begin{pmatrix} 7 & 30 \\ 10 & 43 \end{pmatrix}, \quad \begin{pmatrix} 13 & 16 \\ 30 & 37 \end{pmatrix}, \quad \begin{pmatrix} 5 & 14 \\ 16 & 45 \end{pmatrix}, \quad \begin{pmatrix} 3 & 10 \\ 14 & 47 \end{pmatrix}.$$

The reduced matrices in $\mathcal{Q}(\alpha)$ are

$$\begin{pmatrix} 7 & 10 \\ 30 & 43 \end{pmatrix}, \quad \begin{pmatrix} 13 & 30 \\ 16 & 37 \end{pmatrix}, \quad \begin{pmatrix} 5 & 16 \\ 14 & 45 \end{pmatrix}, \quad \begin{pmatrix} 3 & 14 \\ 10 & 47 \end{pmatrix}.$$

Proof of Theorem 4.3.4. The matrices given in Theorem 4.3.4 are clearly reduced. By Proposition 2.2.29 we have that the matrices in $\mathcal{P}(\alpha)$ are $\mathrm{SL}(2, \mathbb{Z})$ equivalent, and the matrices in $\mathcal{Q}(\alpha)$ are $\mathrm{SL}(2, \mathbb{Z})$ equivalent.

We show that the reduced matrices in $\mathcal{P}$ and $\mathcal{Q}$ are equal if and only if α is evenly palindromic. Let α be an evenly palindromic LLS period. Without loss of generality, assume $\alpha = (a_1, \ldots, a_n)$ is palindromic, so $\alpha = \overline{\alpha}$. Then $\alpha_2 = \overline{\alpha_n}$, where $\alpha_2 = (a_2, \ldots, a_n, a_1)$ and $\alpha_n = (a_n, a_1, \ldots, a_{n-1})$. Similarly, $\alpha_i = \overline{\alpha_{n-i+2}}$ for all $i = 3, \ldots, n$.

Recall that if $\alpha_i = \overline{\alpha_j}$ then $K_1^{n-1}(\alpha_i) = K_2^n(\alpha_j)$. Hence

$$M(\alpha_i) = M(\overline{\alpha_{n-i+2}})^{\intercal} = M(\alpha_{n-i+2})$$

for $i = 2, \ldots, n$. Hence $\mathcal{P} = \mathcal{Q}$.

Assume $\mathcal{P} = \mathcal{Q}$. Then for each $\alpha_i = (b_1, \ldots, b_n)$, with $i = 1, \ldots, n$, we have some $\alpha_j = (c_1, \ldots, c_n)$, with $j \in \{1, \ldots, n\}$, such that

$$\begin{pmatrix} K_2^{n-1}(\alpha_i) & K_2^n(\alpha_i) \\ K_1^{n-1}(\alpha_i) & K_1^n(\alpha_i) \end{pmatrix} = \begin{pmatrix} K_2^{n-1}(\alpha_j) & K_1^{n-1}(\alpha_j) \\ K_2^n(\alpha_j) & K_1^n(\alpha_j) \end{pmatrix}.$$

Then

$$[b_1; b_2 : \ldots : b_n] = \frac{K_1^n(\alpha_i)}{K_2^n(\alpha_i)} = \frac{K_1^n(\alpha_j)}{K_1^{n-1}(\alpha_j)} = [c_1; c_2 : \ldots : c_n],$$

and so $(b_1, \ldots, b_n) = (c_n, \ldots, c_1)$. Hence α is evenly palindromic, and the proof is complete. ∎

4.3.3 General Markov numbers

We introduce general Markov numbers. Let us start with the following definition.

Definition 4.3.5. Let $\mathcal{G}_{\oplus}(\mu, \nu)$ be a Markov graph. For a sequence α in a vertex of $\mathcal{G}_{\oplus}(\mu, \nu)$ define the *general Markov number associated with* α to be

$$\breve{K}(\alpha).$$

The graph $X\big(\mathcal{G}_{\oplus}(\mu, \nu)\big)$ is called a *graph of general Markov numbers.* We denote this graph by $\mathcal{T}(\mu, \nu)$.

We define infinite sequences of regular Markov numbers $\mathcal{L}_v$ and $\mathcal{R}_v$ for a triple $v = (a, Q, b)$ at a vertex of the graph of regular Markov numbers. Recall the ternary operation Σ acting on triples of integers by

$$\Sigma(a, b, c) = 3ab - c.$$

Definition 4.3.6. Define the *regular Markov paths of* v to be $\mathcal{L}_v = (P_i)_{i>0}$ and $\mathcal{R}_v = (Q_i)_{i>0}$, where

$$P_1 = b, \quad Q_1 = a, \quad P_2 = Q_2 = M,$$

and for $i > 2$

$$P_i = \Sigma\big(a, P_{i-1}, P_{i-2}\big),$$
$$Q_i = \Sigma\big(b, Q_{i-1}, Q_{i-2}\big).$$

Example 65. For the vertex $v = (13, 194, 5)$ in $\mathcal{G}_\Sigma(1, 5, 2)$ we have the regular Markov paths

$$\mathcal{L}_v = (13, 194, 7561, 294685, 11485154, 447626321, 17445941365, \ldots),$$
$$\mathcal{R}_v = (5, 194, 2897, 43261, 646018, 9647009, 144059117, 2151239746, \ldots).$$

The definition relies on there being a constant (namely 3) such that both

$$3Qa - b \qquad \text{and} \qquad 3Qb - a$$

are regular Markov numbers. We show using an example that this rule does not follow over to general Markov numbers.

Definition 4.3.7. For a vertex $v = (\alpha, \alpha\beta, \beta)$ in a Markov graph $\mathcal{G}_\oplus(\mu, \nu)$ define the *general Markov paths of* v to be the sequences $\mathcal{L}_v^* = (P_i)_{i>0}$ and $\mathcal{R}_v^* = (Q_i)_{i>0}$, where

$$P_1 = \breve{K}(\beta), \quad Q_1 = \breve{K}(\alpha), \quad P_2 = Q_2 = \breve{K}(\alpha\beta),$$

and for $i > 2$

$$P_i = \breve{K}\big(\alpha^{i-1} \oplus \beta\big),$$
$$Q_i = \breve{K}\big(\alpha \oplus \beta^{i-1}\big).$$

Proposition 4.3.8. *For a vertex v in $\mathcal{G}_\oplus\big((1,1),(2,2)\big)$ we have the following equality of sequences*

$$\mathcal{L}_v^* = \mathcal{L}_v,$$
$$\mathcal{R}_v^* = \mathcal{R}_v.$$

Proof. This is a rewording of the result in Theorem 3.2.7. ∎

Example 66. For the vertex $v = (5, 433, 29)$ in $\mathcal{G}_\Sigma(1, 5, 2)$, the Markov paths are

$$\mathcal{L}_v = (5, 433, 6466, 96557, 1441889, \ldots),$$
$$\mathcal{R}_v = (29, 433, 37666, 3276509, 285018617, \ldots).$$

Note that $1441889 = 3 \cdot 5 \cdot 96557 - 96557$. The recurrence relation is 3 times the first element in $\mathcal{L}_v$.

Consider now the Markov graph $\mathcal{G}_{\oplus}\big((1,1),(3,3)\big)$. The graph of general Markov numbers is $\mathcal{T}\big((1,1),(3,3)\big)$. For the triple $w = (1,18,7)$ at a vertex in $\mathcal{T}\big((1,1),(3,3)\big)$ we have the general Markov paths

$$\begin{aligned}\mathcal{L}^*_w &= (1,18,47,123,322,843,2207,5778,\ldots),\\ \mathcal{R}^*_w &= (7,18,485,13077,352594,9506961,\ldots).\end{aligned}$$

Looking at $\mathcal{L}^*_w$ we see that

$$322 = 3 \cdot 1 \cdot 123 - 47 \quad \text{and} \quad 843 = 3 \cdot 1 \cdot 322 - 123.$$

However, in $\mathcal{R}^*_w$ we have

$$13077 = 27 \cdot 485 - 18 \quad \text{and} \quad 352594 = 27 \cdot 13077 - 485.$$

Since 7 does not divide 27 we have that there is no simple ternary operation to generate $\mathcal{T}\big((1,1),(3,3)\big)$ as there is for $\mathcal{G}_{\Sigma}(1,5,2)$. Note that in $\mathcal{R}^*_w = (Q_i)_{i>0}$, for $i > 2$ we have that $Q_i = 27Q_{i-1} - Q_{i-2}$. We discuss why this is in Chapter 5.

We finish this subsection by relating the graphs of general reduced Markov matrices and general reduced Markov forms with the graph of general Markov numbers. Recall the maps Λ and Υ from Definitions 3.2.35 and 3.2.40.

Theorem 4.3.9. *Let $\mathcal{G}_{\oplus}(\mu,\nu)$ be a Markov graph. Then*

$$\Lambda\big(\mathcal{G}_{\theta}\big(f_\mu, f_\nu\big)\big) = X\big(\mathcal{G}_{\oplus}(\mu,\nu)\big)$$

and

$$\Upsilon\big(\mathcal{G}_{\bullet}\big(M_\mu, M_\nu\big)\big) = X\big(\mathcal{G}_{\oplus}(\mu,\nu)\big).$$

Proof. We have a bijection between the graphs $\mathcal{G}_{\oplus}(\mu,\nu)$ and $\mathcal{G}_{\theta}\big(f_\mu, f_\nu\big)$ connecting the vertices $v = (\alpha,\delta,\beta)$ and $\varphi(v) = \big(f_\alpha, f_\delta, f_\beta\big)$. We show that $\chi(v) = \lambda\big(\varphi(v)\big)$.

$$\begin{aligned}\chi(v) &= \big(\breve{K}(\alpha), \breve{K}(\delta), \breve{K}(\beta)\big)\\ &= \lambda\big(f_\alpha, f_\delta, f_\beta\big)\\ &= \lambda\big(\varphi(v)\big).\end{aligned}$$

Similarly, there is a bijection between $\mathcal{G}_{\oplus}(\mu,\nu)$ and $\mathcal{G}_{\bullet}\big(M_\mu, M_\nu\big)$ connecting the vertices $v = (\alpha,\delta,\beta)$ and $\gamma(v) = \big(M_\alpha, M_\delta, M_\beta\big)$. We show that $\chi(v) = \upsilon\big(\gamma(v)\big)$.

$$\begin{aligned}\chi(v) &= \big(\breve{K}(\alpha), \breve{K}(\delta), \breve{K}(\beta)\big)\\ &= \upsilon\big(M_\alpha, M_\delta, M_\beta\big)\\ &= \upsilon\big(\gamma(v)\big).\end{aligned}$$

This completes the proof. ∎

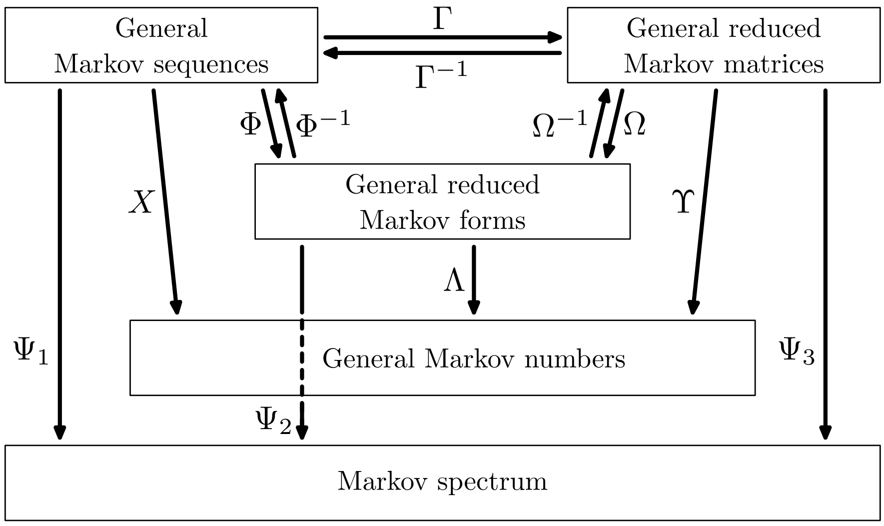

Figure 4.5: The extended structure of Markov theory.

4.4 Maps

We finalise the maps between the various graphs of sequences, matrices, forms, numbers, and the Markov spectrum.

4.4.1 Inverse maps

Recall the definitions for Γ^{-1}, Ω^{-1}, and Φ^{-1} from Chapter 3. From Theorems 3.3.7 and 3.3.14 we have that these are inverse maps to Γ, Ω, and Φ respectively.

4.4.2 Markov spectrum maps

Recall the maps Ψ_i for $i \in \{1, 2, 3\}$ from Chapter 3. We relate the graphs of general Markov matrices, forms, and sequences to the Markov spectrum.

Theorem 4.4.1. *Let $\mathcal{G}_\oplus(\mu, \nu)$ be a Markov graph. Then*

$$\Psi_1\big(\mathcal{G}_\oplus(\mu, \nu)\big) = \Psi_2\big(\mathcal{G}_\theta\big(f_\mu, f_\nu\big)\big)$$

and

$$\Psi_1\big(\mathcal{G}_\oplus(\mu, \nu)\big) = \Psi_3\big(\mathcal{G}_\theta\big(M_\mu, M_\nu\big)\big).$$

Proof. This follows from the definitions of the maps ρ_1, ρ_2, and ρ_3 from Section 3.4.1. ∎

Figures 4.5 and 4.6 show the relations between the different graphs of general Markov theory, and the Markov spectrum.

Recall the result in Proposition 3.4.8. From this and Theorem 4.4.1 we have the proof of the following proposition.

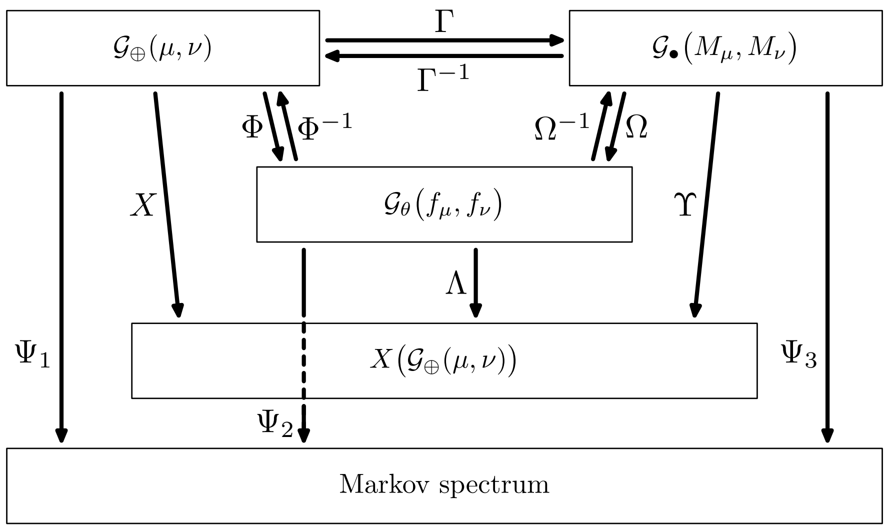

Figure 4.6: The general structure of Markov theory.

Proposition 4.4.2. *Let $\mathcal{G}_{\oplus}(\mu,\nu)$ be a Markov graph with the the lengths of μ and ν even. For every sequence $\alpha = (a_1, \ldots, a_n)$ in every vertex of $\mathcal{G}_{\oplus}(\mu,\nu)$, and the associated form f_α, we have that*

$$\inf_{\mathbb{Z}^2 \setminus \{(0,0)\}} |f_\alpha| = \frac{\sqrt{\Delta(f_\alpha)}}{[\langle\overline{\alpha}\rangle] + [0; \langle\alpha\rangle]}.$$

Proof. Note that $|\alpha| = n$ is even. From Theorem 4.1.14 we have that $m(f_\alpha) = f_\alpha(1,0)$. With this, we need only show that

$$[\langle\overline{\alpha}\rangle] + [0; \langle\alpha\rangle] = \frac{\sqrt{\left(K_1^n(\alpha) + K_2^{n-1}(\alpha)\right)^2 - 4}}{K_1^{n-1}(\alpha)}.$$

But this follows directly from the proof of Proposition 3.4.8. This completes the proof. ■

Remark 4.4.3. Due to the palindromicity requirement of sequences in Markov graphs, there is at least one cyclic shift of δ giving the same value of $|f_\delta|$.

Remark 4.4.4. Currently there is no known way to connect general Markov numbers with the Markov spectrum.

Remark 4.4.5. The major differences between the general and regular cases are that we lose the partial inverses from the forms, matrices, and sequences to the general Markov numbers, and we have no map from general Markov numbers to the spectrum.

Chapter 5

General Markov numbers

In this chapter we study general Markov numbers in more detail.

We start in Section 5.1 where we study general Markov numbers themselves. Recall that for any triple (a, Q, b) of regular Markov numbers there is a recurrence relation generating two infinite sequences of regular Markov numbers. In the general case, there is a recurrence rule for general Markov numbers. However, the recurrence constants for a general triple $\big(\breve{K}(\alpha), \breve{K}(\alpha\beta), \breve{K}(\beta)\big)$ are not always $3\breve{K}(\alpha)$ and $3\breve{K}(\beta)$. They are instead given by the trace of the general reduced Markov matrices associated with $\breve{K}(\alpha)$ and $\breve{K}(\beta)$, as is shown in Theorem 5.1.5.

In Section 5.2 we introduce the uniqueness conjecture for general Markov numbers. We show that in certain cases of general Markov numbers this uniqueness conjecture does not hold.

We close the chapter in Section 5.3, where we mention some open questions in the area, and possible further study.

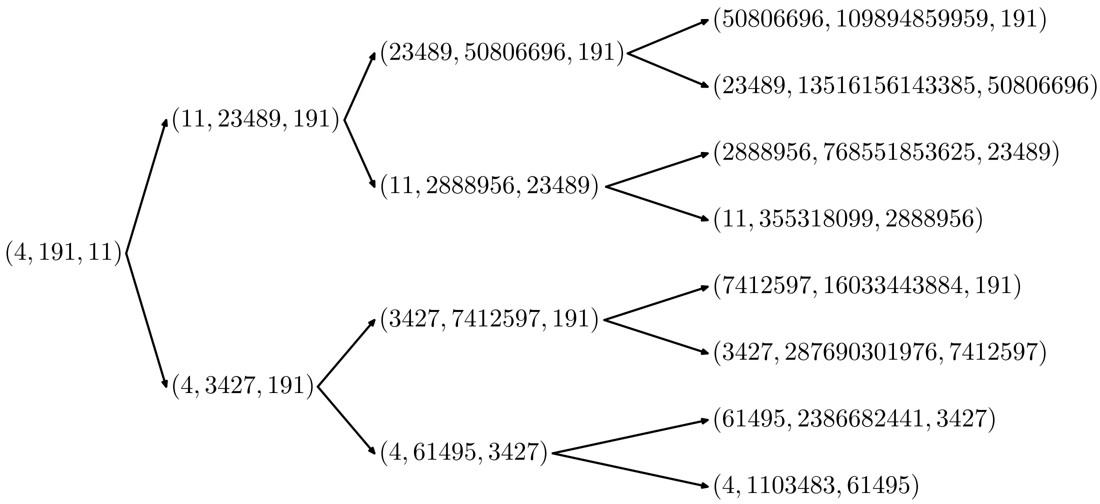

Figure 5.1: The first four levels in the general Markov graphs $\mathcal{T}\big((4,4),(11,11)\big)$

5.1 General Markov numbers

In this section we discuss general Markov numbers in more detail. We discuss a version of the uniqueness conjecture for Markov graphs.

Example 67. Figures 5.1 and 5.2 give the first 4 levels in the general Markov graphs $\mathcal{T}\big((4,4),(11,11)\big)$ and $\mathcal{T}\big((1,1),(3,2,1,1,2,3)\big)$ respectively

Recall the notation for the regular Markov paths $\mathcal{L}_v$ and $\mathcal{R}_v$ from Chapter 3. We define paths for vertices in Markov graphs, and corresponding paths of general Markov numbers.

Definition 5.1.1. Let $v = (\alpha, \alpha\beta, \beta)$ be any vertex in a Markov graph $\mathcal{G}_\oplus(\mu,\nu)$. We define the sequences $\mathcal{L}_\oplus(v)$ and $\mathcal{R}_\oplus(v)$ by

$$\mathcal{L}_\oplus(v) = (\alpha^i\beta)_{i\geq 0},$$
$$\mathcal{R}_\oplus(v) = (\alpha\beta^i)_{i\geq 0}.$$

For the corresponding vertex $\chi(v)$ in the graph of general Markov numbers $\mathcal{T}(\mu,\nu)$ we define the *general Markov paths of* $\chi(v)$ to be

$$\mathcal{L}_\chi(v) = \big(\breve{K}(\alpha^i\beta)\big)_{i\geq 0},$$
$$\mathcal{R}_\chi(v) = \big(\breve{K}(\alpha\beta^i)\big)_{i\geq 0}.$$

Example 68. Let $\alpha = (1,1)$ and $\beta = (5,3,3,5)$, and let $v_1 = (\alpha,\alpha\beta,\beta)$ be a vertex in the Markov graph $\mathcal{G}_\oplus\big((1,1),(5,3,3,5)\big)$. Then the first seven elements of $\mathcal{L}_\chi(v_1)$ are

$$(53, 116, 295, 769, 2012, 5267, 13789),$$

and the first seven elements of $\mathcal{R}_\chi(v_1)$ are

$$(1, 116, 33755, 9822589, 2858339644, 831767013815, 242041342680521).$$

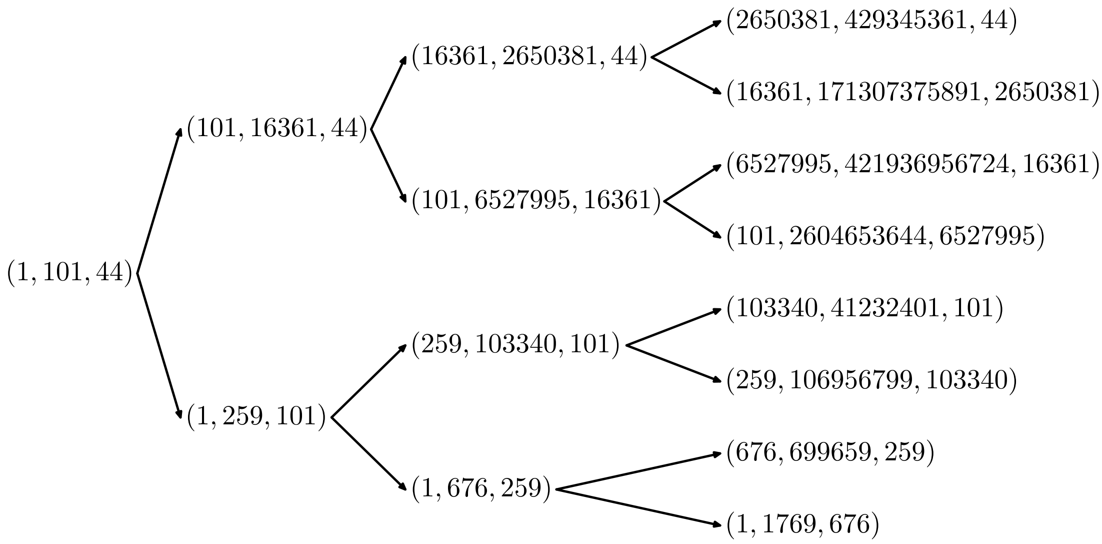

Figure 5.2: The first four levels in the general Markov graph $\mathcal{T}(\mu, \nu)$ for $\mu = (1, 1)$ and $\nu = (3, 2, 1, 1, 2, 3)$

For the vertex $v_2 = (\alpha, \alpha^2\beta, \alpha\beta)$ we have the first seven elements of $\mathcal{L}_\chi(v_2)$ are

$$(116, 295, 769, 2012, 5267, 13789, 36100),$$

and the first seven elements of $\mathcal{R}_\chi(v_2)$ are

$$(1, 295, 200009, 135605807, 91940537137, 62335548573079, 42263409992010425).$$

Note that for any $n > 0$, for the vertices $v_1 = (\alpha, \alpha\beta, \beta)$ and $v_n = (\alpha, \alpha^n\beta, \alpha^{n-1}\beta)$ we have that the general Markov path $\mathcal{L}_\chi(v_n)$ is a subsequence of $\mathcal{L}_\chi(v_1)$.

Remark 5.1.2. By Theorem 3.2.7, we have for vertices v in the graph of regular Markov sequences $\mathcal{G}_\oplus\big((1, 1), (2, 2)\big)$ and $w = \chi(v)$ in $\mathcal{G}_\Sigma(1, 5, 2)$ that

$$\mathcal{L}_\chi(v) = \mathcal{L}_w,$$
$$\mathcal{R}_\chi(v) = \mathcal{R}_w.$$

We show that there is a recurrence relation for the general Markov paths.

Definition 5.1.3. Let $(\alpha, \alpha\beta, \beta)$ be a vertex in a Markov graph $\mathcal{G}_\oplus(\mu, \nu)$, with $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_m)$. We define the *Markov path constants for* $\mathcal{L}_\oplus(v)$ *and* $\mathcal{R}_\oplus(v)$ to be

$$\frac{K_1^{2n-1}(\alpha^2)}{K_1^{n-1}(\alpha)} = \frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)} \qquad \text{and} \qquad \frac{K_1^{2m-1}(\beta^2)}{K_1^{m-1}(\beta)} = \frac{\breve{K}(\beta^2)}{\breve{K}(\beta)}$$

respectively.

Definition 5.1.4. Let $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_m)$, and let $(\alpha, \alpha\beta, \beta)$ be a vertex in a Markov graph $\mathcal{G}_\oplus(\mu, \nu)$. Define the sequences $\mathcal{L}_{\chi,\Sigma}(v) = \big(P_i\big)_{i \geq 1}$ and $\mathcal{R}_{\chi,\Sigma}(v) = \big(Q_i\big)_{i \geq 1}$ for $i \in \{1, 2\}$ by the relations

$$P_1 = \breve{K}(\beta), \quad Q_1 = \breve{K}(\alpha), \quad P_2 = Q_2 = \breve{K}(\alpha\beta),$$

and for $i > 2$ by

$$P_i = \frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)} P_{i-1} - P_{i-2},$$
$$Q_i = \frac{\breve{K}(\beta^2)}{\breve{K}(\beta)} Q_{i-1} - Q_{i-2}.$$

Example 69. Let $\alpha = (1,1)$ and $\beta = (5,3,3,5)$, and let $v = (\alpha, \alpha\beta, \beta)$ be a vertex in the Markov graph $\mathcal{G}_\oplus\big((1,1),(5,3,3,5)\big)$. We have that $\breve{K}(\alpha) = 1$, $\breve{K}(\beta) = 53$, and $\breve{K}(\alpha\beta) = 116$. The Markov path constant for $\mathcal{L}_\oplus(v)$ is

$$\frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)} = \frac{3}{1},$$

and the Markov path constant for $\mathcal{R}_\oplus(v)$ is

$$\frac{\breve{K}(\beta^2)}{\breve{K}(\beta)} = \frac{15423}{53} = 291.$$

Note that $3 \cdot 116 - 53 = 295$ and $291 \cdot 116 - 1 = 33755$. Following on in this way we have that the first seven elements in $\mathcal{L}_{\chi,\Sigma}(v)$ are

$$(53, 116, 295, 769, 2012, 5267, 13789),$$

while the first seven elements in $\mathcal{R}_{\chi,\Sigma}(v)$ are

$$(1, 116, 33755, 9822589, 2858339644, 831767013815, 242041342680521).$$

Note that the first seven elements of $\mathcal{L}_{\chi,\Sigma}(v)$ and $\mathcal{L}_\chi(v)$ are the same. Also, the first seven elements of $\mathcal{R}_{\chi,\Sigma}(v)$ and $\mathcal{R}_\chi(v)$ are the same.

We show that the general Markov paths are given by the recurrence relations in Definition 5.1.4.

Theorem 5.1.5. *Let α, λ, and ρ be the following sequences of positive integers*

$$\alpha = (a_1, \dots, a_n),$$
$$\lambda = (b_1, \dots, b_l),$$
$$\rho = (c_1, \dots, c_r),$$

with n, l, and r positive integers, and n even. Then we have that

$$\frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)} = \frac{\breve{K}(\lambda\alpha^2\rho) + \breve{K}(\lambda\rho)}{\breve{K}(\lambda\alpha\rho)}.$$

Corollary 5.1.6. For a vertex v in a Markov graph $\mathcal{G}_\oplus(\mu, \nu)$ we have that

$$\mathcal{L}_{\chi,\Sigma}(v) = \mathcal{L}_\chi(v),$$
$$\mathcal{R}_{\chi,\Sigma}(v) = \mathcal{R}_\chi(v).$$

Example 70. The triple $v = \big((1,1),(1,1,3,2,1,1,2,3),(3,2,1,1,2,3)\big)$ is at a vertex in the Markov graph $\mathcal{G}_{\oplus}\big((1,1),(3,2,1,1,2,3)\big)$. The corresponding vertex in $\mathcal{T}\big((1,1),(3,2,1,1,2,3)\big)$ is

$$\chi(v) = (1, 101, 44).$$

Theorem 5.1.5 says that the central elements of the vertices in these paths are given by recurrence relations with the Markov path constants

$$\frac{\breve{K}(1,1,1,1)}{\breve{K}(1,1)} = 3\cdot 1 \qquad \text{and} \qquad \frac{\breve{K}(3,2,1,1,2,3,3,2,1,1,2,3)}{\breve{K}(3,2,1,1,2,3)} = 162.$$

We have that the first 3 elements in both $\mathcal{R}_{\chi,\Sigma}(v)$ and $\mathcal{R}_{\chi}(v)$ are

$$(1, 101, 16361).$$

The fourth and fifth elements in $\mathcal{R}_{\chi,\Sigma}(v)$ are

$$162\cdot 16361 - 101 = 2650381 \qquad \text{and} \qquad 162\cdot 2650381 - 16361 = 429345361,$$

and the fourth and fifth elements in $\mathcal{R}_{\chi}(v)$ are

$$\begin{aligned}&\breve{K}(1,1,3,2,1,1,2,3,3,2,1,1,2,3,3,2,1,1,2,3) = 2650381, \quad \text{and}\\ &\breve{K}(1,1,3,2,1,1,2,3,3,2,1,1,2,3,3,2,1,1,2,3,3,2,1,1,2,3) = 429345361.\end{aligned}$$

We first have the following lemma.

Lemma 5.1.7. Let $\alpha = (a_1,\dots,a_n)$ be a sequence in some vertex of a Markov graph $\mathcal{G}_{\oplus}(\mu,\nu)$. Then

$$\frac{K_1^{2n-1}(\alpha^2)}{K_1^{n-1}(\alpha)} = K_1^n(\alpha) + K_2^{n-1}(\alpha).$$

Proof. Splitting the left hand side of the equation by the formula in Proposition 2.1.6 we get

$$\begin{aligned}\frac{K_1^{2n-1}(\alpha^2)}{K_1^{n-1}(\alpha)} &= \frac{K_1^n(\alpha)K_1^{n-1}(\alpha) + K_1^{n-1}(\alpha)K_2^{n-1}(\alpha)}{K_1^{n-1}(\alpha)}\\ &= K_1^n(\alpha) + K_2^{n-1}(\alpha).\end{aligned}$$

This concludes the proof. ∎

Remark 5.1.8. Note that the value $K_1^n(\alpha)+K_2^{n-1}(\alpha)$ is the trace of the associated general reduced Markov matrix M_α.

Proof of Theorem 5.1.5. From Lemma 5.1.7, the equation in Theorem 5.1.5 is equivalent to

$$K_1^n(\alpha) + K_2^{n-1}(\alpha) = \frac{\breve{K}(\lambda\alpha^2\rho) + \breve{K}(\lambda\rho)}{\breve{K}(\lambda\alpha\rho)}.$$

We investigate the quantity

$$T_R = \breve{K}(\lambda\alpha\rho)K_1^n(\alpha) + \breve{K}(\lambda\alpha\rho)K_2^{n-1}(\alpha) - \breve{K}(\lambda\alpha^2\rho) - \breve{K}(\lambda\rho). \tag{5.1}$$

We show that $T_R = 0$. We use the following relations in our proof, coming from Proposition 2.1.6:

$$\breve{K}(\lambda\alpha^2\rho) = K_1^{l+n}(\lambda\alpha)\breve{K}(\alpha\rho) + \breve{K}(\lambda\alpha)K_2^{n+r-1}(\alpha\rho); \tag{5.2}$$

$$\breve{K}(\lambda\alpha\rho) = K_1^{l}(\lambda)\breve{K}(\alpha\rho) + \breve{K}(\lambda)K_2^{n+r-1}(\alpha\rho); \tag{5.3}$$

$$\breve{K}(\lambda\alpha) = K_1^{l}(\lambda)\breve{K}(\alpha) + \breve{K}(\lambda)K_2^{n-1}(\alpha); \tag{5.4}$$

$$K_1^{l+n}(\lambda\alpha) = K_1^{l}(\lambda)K_1^n(\alpha) + \breve{K}(\lambda)K_2^{n}(\alpha); \tag{5.5}$$

Substituting Equations (5.2), (5.3), and (5.4), into Equation (5.1) we get

$$\begin{aligned} T_R =&\breve{K}(\lambda\alpha\rho)K_1^n(\alpha) + \Big(K_1^l(\lambda)\breve{K}(\alpha\rho) + \breve{K}(\lambda)K_2^{n+r-1}(\alpha\rho)\Big)K_2^{n-1}(\alpha) - \\ &\Big(K_1^{l+n}(\lambda\alpha)\breve{K}(\alpha\rho) + \Big(K_1^l(\lambda)\breve{K}(\alpha) + \breve{K}(\lambda)K_2^{n-1}(\alpha)\Big)K_2^{n+r-1}(\alpha\rho)\Big) - \breve{K}(\lambda\rho). \end{aligned}$$

After cancelling reciprocal terms we have

$$\begin{aligned} T_R =&\breve{K}(\lambda\alpha\rho)K_1^n(\alpha) + K_1^l(\lambda)\breve{K}(\alpha\rho)K_2^{n-1}(\alpha) - \\ &K_1^{l+n}(\lambda\alpha)\breve{K}(\alpha\rho) - K_1^l(\lambda)\breve{K}(\alpha)K_2^{n+r-1}(\alpha\rho) - \breve{K}(\lambda\rho). \end{aligned} \tag{5.6}$$

Now substituting Equations (5.3) and (5.5) to Equation (5.6) we obtain

$$\begin{aligned} T_R =&\Big(K_1^l(\lambda)\breve{K}(\alpha\rho) + \breve{K}(\lambda)K_2^{n+r-1}(\alpha\rho)\Big)K_1^n(\alpha) + K_1^l(\lambda)\breve{K}(\alpha\rho)K_2^{n-1}(\alpha) - \\ &\Big(K_1^l(\lambda)K_1^n(\alpha) + \breve{K}(\lambda)K_2^n(\alpha)\Big)\breve{K}(\alpha\rho) - K_1^l(\lambda)\breve{K}(\alpha)K_2^{n+r-1}(\alpha\rho) - \breve{K}(\lambda\rho). \end{aligned} \tag{5.7}$$

Again cancelling reciprocal terms we have

$$\begin{aligned} T_R =&\breve{K}(\lambda)K_2^{n+r-1}(\alpha\rho)K_1^n(\alpha) + K_1^l(\lambda)\breve{K}(\alpha\rho)K_2^{n-1}(\alpha) - \\ &\breve{K}(\lambda)K_2^n(\alpha)\breve{K}(\alpha\rho) - K_1^l(\lambda)\breve{K}(\alpha)K_2^{n+r-1}(\alpha\rho) - \breve{K}(\lambda\rho). \end{aligned}$$

After factorising this equation becomes

$$\begin{aligned} T_R =&\breve{K}(\lambda)\Big(K_2^{n+r-1}(\alpha\rho)K_1^n(\alpha) - K_2^n(\alpha)\breve{K}(\alpha\rho)\Big) + \\ &K_1^l(\lambda)\Big(\breve{K}(\alpha\rho)K_2^{n-1}(\alpha) - \breve{K}(\alpha)K_2^{n+r-1}(\alpha\rho)\Big) - \breve{K}(\lambda\rho). \end{aligned} \tag{5.8}$$

Let us study the expressions in the brackets of this equation. We split the continuants using the result in Proposition 2.1.6. First of all, we have

$$\begin{aligned} K_2^{n+r-1}(\alpha\rho)K_1^n(\alpha) - K_2^n(\alpha)\breve{K}(\alpha\rho) =&\Big(K_2^n(\alpha)\breve{K}(\rho) + K_2^{n-1}(\alpha)K_2^{r-1}(\rho)\Big)K_1^n(\alpha) \\ &- K_2^n(\alpha)\Big(K_1^n(\alpha)\breve{K}(\rho) + \breve{K}(\alpha)K_2^{r-1}(\rho)\Big) \\ =&\big(K_1^n(\alpha)K_2^{n-1}(\alpha) - K_2^n(\alpha)\breve{K}(\alpha)\big)K_2^{r-1}(\rho) \\ =&K_2^{r-1}(\rho), \end{aligned}$$

with the last equality holding since n is even. Secondly, it holds that

$$\begin{aligned}\breve{K}(\alpha\rho)K_2^{n-1}(\alpha) - \breve{K}(\alpha)K_2^{n+r-1}(\alpha\rho) =&\Big(K_1^n(\alpha)\breve{K}(\rho) + \breve{K}(\alpha)K_2^{r-1}(\rho)\Big)K_2^{n-1}(\alpha)\\ &- \breve{K}(\alpha)\Big(K_2^n(\alpha)\breve{K}(\rho) + K_2^{n-1}(\alpha)K_2^{r-1}(\rho)\Big)\\ =&\Big(K_1^n(\alpha)K_2^{n-1}(\alpha) - \breve{K}(\alpha)K_2^n(\alpha)\Big)\breve{K}(\rho)\\ =&\breve{K}(\rho).\end{aligned}$$

Using this, Equation (5.8) becomes

$$T_R = K_1^l(\lambda)\breve{K}(\rho) + \breve{K}(\lambda)K_2^{r-1}(\rho) - \breve{K}(\lambda\rho),$$

and so $T_R = 0$, by Proposition 2.1.6. ■

Remark 5.1.9. For general Markov numbers one must know the general Markov sequences associated with a vertex of numbers before generating the recurrence relation (which is why the graph needs to be generated from a Markov graph), whereas for regular Markov numbers this is not the case.

5.2 Uniqueness conjecture

In this short section we recall the uniqueness conjecture for regular Markov numbers. We give an analogous conjecture for general Markov numbers. We show that the conjecture fails for certain graphs of general Markov numbers in Theorem 5.2.8.

Recall the uniqueness conjecture for regular Markov numbers.

Conjecture 5.2.1 (Regular uniqueness conjecture (Frobenius [38]))**.** In the graph $\mathcal{T}\big((1,1),(2,2)\big)$ each regular Markov number appears as the maximal element in exactly one vertex.

This conjecture is well studied, and shows up in many interesting areas. We refer to the book by M. Aigner [2] for a general reference.

Conjecture 5.2.2 (General uniqueness conjecture)**.** In the graph of general Markov numbers $T(\mu,\nu)$ each general Markov number appears as the maximal element in exactly one vertex.

We present a proposition that gives a condition for whether the general uniqueness conjecture fails for a Markov graph $\mathcal{T}(\mu,\nu)$.

Proposition 5.2.3. *Let $v=(\alpha,\alpha\beta,\beta)$ be a vertex in a Markov graph $\mathcal{G}_\oplus(\mu,\nu)$. Then $\chi(v)=(a,Q,b)$ is a vertex in the graph of general Markov numbers $\mathcal{T}(\mu,\nu)$. Let $\mathcal{L}_\chi(v)=\big(P_i\big)_{i\geq 0}$ and $\mathcal{R}_\chi(v)=\big(Q_i\big)_{i\geq 0}$.*

Assume there exist positive integers $j>1$ and $k>1$ such that $P_j=Q_k$. Then there are distinct vertices

$$\big(a,P_j,P_{j-1}\big) \qquad \textit{and} \qquad \big(Q_{k-1},Q_k,b\big)$$

in the graph $X\big(\mathcal{G}_\oplus(\mu,\nu)\big)$ with the same largest element. Hence the uniqueness conjecture for $\mathcal{T}(\mu,\nu)$ fails.

The following lemma ensures that the considered P_j and Q_k are the largest elements in their respective vertices.

Lemma 5.2.4. The sequences $\mathcal{L}_\chi(v)=\big(P_i\big)_{i\geq 0}$ and $\mathcal{R}_\chi(v)=\big(Q_i\big)_{i\geq 0}$ are strictly increasing.

Proof. By inspection we know that $\breve{K}(\alpha)<\breve{K}(\alpha\beta)$ and $\breve{K}(\beta)<\breve{K}(\alpha\beta)$. Let the Markov sequence for some element $P_i\in\mathcal{L}_\chi(v)$ be $\delta=(x_1,\ldots,x_d)$. Then

$$\begin{aligned}P_{i+1}&=\breve{K}(\alpha\delta)\\&=K_1^n(\alpha)\breve{K}(\delta)+\breve{K}(\alpha)K_2^d(\delta)\\&>K_1^n(\alpha)\breve{K}(\delta)\geq\breve{K}(\delta).\end{aligned}$$

The strict inequality is due to the fact that continuants of positive integers are always positive.

A similar argument holds for elements of $Q_i \in \mathcal{R}_\chi(v)$. The proof follows. ∎

The next lemma says that the triples considered in Proposition 5.2.3 are in fact vertices in a graph of general Markov numbers.

Lemma 5.2.5. Let $v = (\alpha, \alpha\beta, \beta)$ be a vertex in a Markov graph $\mathcal{G}_\oplus(\mu, \nu)$. Then $\chi(v) = (a, Q, b)$ is a vertex in the graph of general Markov numbers $\mathcal{T}(\mu, \nu)$. Let $\mathcal{L}_\chi(v) = \big(P_i\big)_{i \geq 0}$ and $\mathcal{R}_\chi(v) = \big(Q_i\big)_{i \geq 0}$. Then the triples

$$\big(a, P_i, P_{i-1}\big) \qquad \text{and} \qquad \big(Q_{i-1}, Q_i, b\big)$$

are both vertices in the graph $X\big(\mathcal{G}_\oplus(\mu, \nu)\big)$, for all $i > 0$.

Proof. This follows from Corollary 5.1.6, and the definition of the paths $\mathcal{L}_\chi(v)$ and $\mathcal{R}_\chi(v)$. ∎

Proof of Proposition 5.2.3. The proof follows from the previous two lemmas. ∎

5.2.1 First counterexamples to the general uniqueness conjecture

We define certain graphs of general Markov numbers for which the uniqueness conjecture is false.

Definition 5.2.6. For any positive integer n define positive integers a_n and b_n by

$$a_n = n^2 + 3,$$
$$b_n = n^4 + 5n^2 + 5.$$

Note that $\gcd(a_n, b_n) = 1$, and that the ratio b_n/a_n is equal to the continued fraction

$$\frac{b_n}{a_n} = \left[1 + n^2; 1 : 2 + n^2\right].$$

Definition 5.2.7. Define the sequences $S_n(0)$ and $S_n(1)$ by

$$S_n(0) = (na_n, na_n), \quad S_n(1) = (nb_n, nb_n).$$

We denote the graphs of general Markov numbers made from these sequences by $\mathcal{T}_n = \mathcal{T}\big(S_n(0), S_n(1)\big)$.

We present counter examples to the general uniqueness conjecture.

Theorem 5.2.8. *The uniqueness conjecture for general Markov numbers does not hold for any graph $\mathcal{T}_n$, where n is a positive integer.*

In the graph of general Markov numbers $\mathcal{T}_n$ for any $n > 0$ there are triples defined for all $j \geq 1$ by

$$\Big(\breve{K}\big(S_n(0)\big),\quad \breve{K}\big(S_n(0)^{5j+1}S_n(1)\big),\quad \breve{K}\big(S_n(0)^{5j}S_n(1)\big)\Big),$$
$$\Big(\breve{K}\big(S_n(0)S_n(1)^{3j}\big),\quad \breve{K}\big(S_n(0)S_n(1)^{3j+1}\big),\quad \breve{K}\big(S_n(1)\big)\Big).$$

We show that the largest element of these triples are equal, and hence the uniqueness conjecture for general Markov numbers fails for the graphs $\mathcal{T}_n$. More specifically we have the following proposition.

Proposition 5.2.9. *For all positive integers n and j we have that*

$$\breve{K}\big(S_n(0)^{5j+1}S_n(1)\big) = \breve{K}\big(S_n(0)S_n(1)^{3j+1}\big).$$

Example 71. The sequences $S_n(0)$ and $S_n(1)$ are given for $n = 1, \ldots, 4$ in the following table.

n	C_n	$S_n(0)$	$S_n(1)$
1	$[2,1,3]$	$(4,4)$	$(11,11)$
2	$[5,1,6]$	$(14,14)$	$(82,82)$
3	$[10,1,11]$	$(36,36)$	$(393,393)$
4	$[17,1,18]$	$(76,76)$	$(1364,1364)$

The simplest examples are in the graph $\mathcal{T}_1$, which contains the triples

$$(4, 355318099,\ 19801199),$$
$$(2888956, 355318099,\ 11).$$

5.2.2 Proof of Theorem 5.2.8

Theorem 5.2.8 follows from Proposition 5.2.9. To prove this proposition we first define sequences of positive integers $\big(L_n(j)\big)_{j>0}$ and $\big(R_n(j)\big)_{j>0}$ containing the values

$$\breve{K}\big(S_n(0)^{5j+1}S_n(1)\big) \quad \text{and} \quad \breve{K}\big(S_n(0)S_n(1)^{3j+1}\big).$$

We show in Lemmas 5.2.14 and 5.2.15 that both $\big(L_n(j)\big)_{j>0}$ and $\big(R_n(j)\big)_{j>0}$ are subsequences of another sequence $\big(A_n(j)\big)_{j>0}$ for every $n > 0$. Then we show that their elements align within $\big(A_n(j)\big)_{j>0}$ in such a way that Proposition 5.2.9 holds.

Definition 5.2.10. Let n be a positive integer, $a_n = n^2+3$, and $b_n = n^4+5n^2+5$. Let

$$l_n = (na_n)^2 + 2, \quad r_n = (nb_n)^2 + 2,$$
$$L_n(1) = \breve{K}(na_n, na_n, nb_n, nb_n), \quad L_n(2) = \breve{K}(na_n, na_n, na_n, na_n, nb_n, nb_n),$$
$$R_n(1) = \breve{K}(na_n, na_n, nb_n, nb_n), \quad R_n(2) = \breve{K}(na_n, na_n, nb_n, nb_n, nb_n, nb_n).$$

For $j > 2$ let

$$L_n(j) = l_nL_n(j-1) - L_n(j-2) \quad \text{and} \quad R_n(j) = r_nR_n(j-1) - R_n(j-2).$$

We relate the sequences $\big(L_n(j)\big)_{j>0}$ and $\big(R_n(j)\big)_{j>0}$ to the numbers in Theorem 5.2.8.

Proposition 5.2.11. *The following statements are equivalent:*

(*i*) *For all positive integers n and i we have that*

$$\breve{K}\big(S_n(0)^{5i+1} \oplus S_n(1)\big) = \breve{K}\big(S_n(0) \oplus S_n(1)^{3i+1}\big).$$

(*ii*) *For all positive integers n and i we have that*

$$L_n(5i+1) = R_n(3i+1).$$

Proof. Let $\alpha = (na_n, na_n)$ and $\beta = (nb_n, nb_n)$. Note that

$$\frac{\breve{K}(\alpha^2)}{\breve{K}(\alpha)} = \frac{\breve{K}(na_n, na_n, na_n, na_n)}{\breve{K}(na_n, na_n)} = \frac{na_n\big((na_n)^2+1\big)+na_n}{na_n} = (na_n)^2+2 = l_n,$$
$$\frac{\breve{K}(\beta^2)}{\breve{K}(\beta)} = \frac{\breve{K}(nb_n, nb_n, nb_n, nb_n)}{\breve{K}(nb_n, nb_n)} = \frac{nb_n\big((nb_n)^2+1\big)+nb_n}{nb_n} = (nb_n)^2+2 = r_n,$$

and

$$\begin{aligned} L_n(1) = R_n(1) &= \breve{K}(\alpha\beta), \\ L_n(2) &= \breve{K}(\alpha^2\beta), \\ R_n(2) &= \breve{K}(\alpha\beta^2). \end{aligned}$$

From Theorem 5.1.5 we have, by induction, that

$$\begin{aligned} L_n(j) &= \breve{K}\big(S_n(0)^j \oplus S_n(1)\big), \\ R_n(j) &= \breve{K}\big(S_n(0) \oplus S_n(1)^j\big). \end{aligned}$$

This completes the proof. ■

Example 72. The first 6 elements of the sequences $\big(L_1(j)\big)_{j>0}$ and $\big(R_1(j)\big)_{j>0}$ are

$$\big(L_1(j)\big)_{j=1}^{6} = (191, 3427, 61495, 1103483, 19801199, 355318099),$$
$$\big(R_1(j)\big)_{j=1}^{6} = (191, 23489, 2888956, 355318099, 43701237221, 5374896860084).$$

Next we define the sequence $\big(A_n(j)\big)_{j>0}$.

Definition 5.2.12. Let $A_n(1) = 1$ and $A_n(2) = n(n^2+4)$. Then for $j > 2$ define

$$A_n(j) = nA_n(j-1) + A_n(j-2).$$

Remark 5.2.13. We guessed the structure of this sequence from looking at the case for $n = 1$, where $\big(A_1(j)\big)_{j>0}$ is the sequence A022095 in [87].

We show that $\big(R_1(j)\big)_{j>0}$ is a subsequence of $\big(A_1(j)\big)_{j>0}$.

Lemma 5.2.14. For all positive integers n and j we have that

$$R_n(j) = A_n(10j).$$

Proof. For any $n > 0$ we have $na_n = 3n + n^3$, $nb_n = n^5 + 5n^3 + 5n$, and also that $r_n = (nb_n)^2 + 2$. Hence

$$R_n(1) = \breve{K}(na_n, na_n, nb_n, nb_n) = n^{11} + 11n^9 + 44n^7 + 76n^5 + 51n^3 + 8n.$$

By computation we see that $A_n(10) = R_n(1)$. Also we see that

$$\begin{aligned} R_n(2) =& n^{21} + 21\,n^{19} + 189\,n^{17} + 951\,n^{15} + 2926\,n^{13} + 5655\,n^{11} + \\ & 6787\,n^9 + 4818\,n^7 + 1827\,n^5 + 301\,n^3 + 13\,n = A_n(20). \end{aligned}$$

This serves as a base of induction.

Assume that $A_n(10j) = R_n(j)$ for all $j = 1, \ldots, k-1$, for some $k > 2$. Then

$$\begin{aligned} R_n(k) &= r_n R_n(k-1) - R_n(k-2) \\ &= r_n A_n\big(10(k-1)\big) - A_n\big(10(k-2)\big), \\ &= r_n A_n(10k-10) - A_n(10k-20), \end{aligned}$$

with the first equality following definition, and the second equality following the induction hypothesis. Note that

$$\begin{aligned} A_n(10k-10) &= nA_n(10k-10-1) + A_n(10k-10-2) \\ &= (n^2+1)A_n(10k-10-2) + A_n(10k-10-3) \\ &\;\vdots \\ &= x_1 A_n(10k-10-9) + x_2 A_n(10k-10-10), \end{aligned}$$

where x_1 and x_2 are given through direct computation (we used Maple software) by

$$\begin{aligned} x_1 &= n^9 + 8\,n^7 + 21\,n^5 + 20\,n^3 + 5\,n, \\ x_2 &= n^8 + 7\,n^6 + 15\,n^4 + 10\,n^2 + 1. \end{aligned}$$

In the same manner we have

$$\begin{aligned} A_n(10k) &= nA_n(10k-1) + A_n(10k-2) \\ &= (n^2+1)A_n(10k-2) + A_n(10k-3) \\ &\;\vdots \\ &= y_1 A_n(10k-19) + y_2 A_n(10k-20), \end{aligned}$$

where y_1 and y_2 are given through direct computation by

$$\begin{aligned} y_1 =& n^{19} + 18\,n^{17} + 136\,n^{15} + 560\,n^{13} + 1365\,n^{11} + \\ & 2002\,n^9 + 1716\,n^7 + 792\,n^5 + 165\,n^3 + 10\,n, \\ y_2 =& n^{18} + 17\,n^{16} + 120\,n^{14} + 455\,n^{12} + 1001\,n^{10} + \\ & 1287\,n^8 + 924\,n^6 + 330\,n^4 + 45\,n^2 + 1. \end{aligned}$$

Using this information we have that

$$\begin{aligned} R_n(k) &= r_n A_n(10k-10) - A_n(10k-20) \\ &= r_n\big(x_1 A_n(10k-10-9) + x_2 A_n(10k-10-10)\big) - A_n(10k-20) \\ &= r_n x_1 A_n(10k-19) + (r_n x_2 - 1) A_n(10k-20). \end{aligned}$$

Through direct computation we see that $y_1 = r_n x_1$ and $y_2 = r_n x_2 - 1$. Hence we have that

$$R_n(k) = A_n(10k),$$

so the induction holds and the proof is complete. ■

Next we show that $\big(L_1(j)\big)_{j>0}$ is a subsequence of $\big(A_1(j)\big)_{j>0}$.

Lemma 5.2.15. For all positive integers n and i we have that

$$L_n(j) = A_n\big(10 + 6(j-1)\big).$$

Proof. We use induction. We have that $A_n(10) = L_n(1)$, as in the proof for Lemma 5.2.14 . We also have $l_n = (n^3+3n)^2{+}2$, and that

$$\begin{aligned} A_n(16) &= n^{17} + 17n^{15} + 119n^{13} + 441n^{11} + 925n^9 + 1086n^7 + 658n^5 + 169n^3 + 11n \\ &= L_n(2). \end{aligned}$$

This is our base of induction. Assume that $A_n\big(10{+}6(j{-}1)\big) = A_n(6j{+}4) = L_n(j)$ for all $j = 1, \ldots, k{-}1$, for some $k > 2$. Then

$$\begin{aligned} L_n(k) &= l_n L_n(k-1) - L_n(k-2) \\ &= l_n A_n(6k-2) - A_n(6k-8), \end{aligned}$$

with the first equality following definition, and the second equality following the induction hypothesis. In a similar way to the proof for $\big(R_n(j)\big)_{j>0} = \big(A_n(j)\big)_{j>0}$ we have that

$$A_n(6k-2) = w_1 A_n(6k-7) + w_2 A_n(6k-8),$$

where $w_1 = n^5 + 4n^3 + 3n$ and $w_2 = n^4 + 3n^2 + 1$. Also we have

$$A_n\big(10 + 6(k-1)\big) = A_n(6k+4) = z_1 A_n(6k-7) + z_2 A_n(6k-8),$$

where

$$\begin{aligned} z_1 &= n^{11} + 10\,n^9 + 36\,n^7 + 56\,n^5 + 35\,n^3 + 6\,n, \\ z_2 &= n^{10} + 9\,n^8 + 28\,n^6 + 35\,n^4 + 15\,n^2 + 1. \end{aligned}$$

Now we have that

$$\begin{aligned} L_n(k) &= l_n A_n(6k-2) - A_n(6k-8), \\ &= l_n w_1 A_n(6k-7) + (l_n w_2 - 1) A_n(6k-8), \end{aligned}$$

and by direct computation we see that $z_1 = l_n w_1$ and $z_2 = l_n w_2 - 1$.

So $L_n(k) = A_n\big(10 + 6(k-1)\big)$ and induction holds. This completes the proof. ■

We prove Proposition 5.2.9.

Proof of Proposition 5.2.9. Given Lemmas 5.2.14 and 5.2.15 we need only show the values $L_n(5j+1)$ and $R_n(3j+1)$ align within the sequence $\big(A_n(j)\big)_{j>0}$. Indeed we have that

$$\begin{aligned} L_n(5j+1) &= A_n\big(10+6(5j+1-1)\big) = A_n(30j+10), \\ R_n(3j+1) &= A_n\big(10(3j+1)\big) = A_n(30j+10). \end{aligned}$$

Hence the claim is proved. ■

Theorem 5.2.8 follows as a corollary.

5.3 Further study

In this final section we detail some current problems in the area of Markov numbers.

In Subsection 5.3.1 we state some experimental evidence for cases of the general uniqueness conjecture not covered in Theorem 5.2.8.

In Section 5.3.2 we list some problems of interest, that have no current solutions.

5.3.1 Some experimental data for the general uniqueness conjecture

Here we list the Markov graphs $\mathcal{T}(\mu,\nu)$ for which we have checked the general uniqueness conjecture computationally, and found no counter examples. The sequences have been checked to satisfy the conditions of a Markov graph from Definition 4.1.4.

For the following sequences μ and ν we have checked the general uniqueness conjecture for the first 14 levels of the graph $\mathcal{T}(\mu,\nu)$ and found no counterexamples.

$$
\begin{aligned}
&\mu=(11,i,i,11) \quad \text{and} \quad \nu=(4,j,j,4), \quad \text{for } 1\le i,j\le 4\\
&\mu=(i,11,11,i) \quad \text{and} \quad \nu=(j,4,4,j), \quad \text{for } 1\le i,j\le 4\\
&\mu=(11,11,11,11) \quad \text{and} \quad \nu=(4,4,4,4),\\
&\mu=(11,1,1,1,1,11) \quad \text{and} \quad \nu=(4,1,1,1,1,4),\\
&\mu=(11,1,2,2,1,11) \quad \text{and} \quad \nu=(4,1,2,2,1,4),\\
&\mu=(11,2,2,2,2,11) \quad \text{and} \quad \nu=(4,2,2,2,2,4),\\
&\mu=(11,2,1,1,2,11) \quad \text{and} \quad \nu=(4,2,1,1,2,4),\\
&\mu=(11,11,1,1,11,11) \quad \text{and} \quad \nu=(4,4,1,1,4,4)\\
&\mu=(11,11,1,1,1,1,11,11) \quad \text{and} \quad \nu=(4,4,1,1,1,1,4,4)\\
&\mu=(11,11,1,2,2,1,11,11) \quad \text{and} \quad \nu=(4,4,1,2,2,1,4,4)\\
&\mu=(11,11,2,2,2,2,11,11) \quad \text{and} \quad \nu=(4,4,2,2,2,2,4,4)\\
&\mu=(11,11,2,1,1,2,11,11) \quad \text{and} \quad \nu=(4,4,2,1,1,2,4,4).
\end{aligned}
$$

5.3.2 Open Questions

In this subsection we mention some interesting open problems in the area of Markov numbers that may be studied further.

Markov sequence limits

For a vertex $(\alpha, \alpha\beta, \beta)$ in a Markov tree we have the sequences

$$\mathcal{L}_{\oplus}(v) = (\alpha^i\beta)_{i\geq 0},$$
$$\mathcal{R}_{\oplus}(v) = (\alpha\beta^i)_{i\geq 0}.$$

We are interested in the limits of the Markov values of these sequences. That is, for a sequence δ in either $\mathcal{L}_{\oplus}(v)$ or $\mathcal{R}_{\oplus}(v)$ has Markov value

$$\big[\langle\overline{\delta}\rangle\big] + [0; \langle\delta\rangle].$$

The limit of the Markov values in $\mathcal{L}_{\oplus}(v)$ is

$$\lim\big(\alpha^i\beta\big) = \lim_{i\to\infty}\big[\langle\overline{\beta}\overline{\alpha}^i\rangle\big] + [0; \langle\alpha^i\beta\rangle] = \big[\overline{\beta}; \langle\overline{\alpha}\rangle\big] + [0; \langle\alpha\rangle],$$

and the limit of the Markov values in $\mathcal{R}_{\oplus}(v)$ is

$$\lim\big(\alpha\beta^i\big) = \lim_{i\to\infty}\big[\langle\overline{\beta}^i\overline{\alpha}\rangle\big] + [0; \langle\alpha\beta^i\rangle] = \big[\langle\overline{\beta}\rangle\big] + [0; \alpha; \langle\beta\rangle].$$

In the graph $\mathcal{G}_{\oplus}\big((1,1),(2,2)\big)$ these limits always equal 3. However, in other graphs this is not the case.

Example 73. In the graph $\mathcal{G}_{\oplus}\big((1,1),(3,3)\big)$ let $\alpha = (1,1,3,3)$ and $\beta = (3,3)$. Then the limits $\lim\big(\alpha^i\beta\big)$ and $\lim\big(\alpha\beta^i\big)$ are given by

$$\lim\big(\alpha^i\beta\big) = \big[(3,3); \langle(3,3,1,1)\rangle\big] + [0; \langle 1,1,3,3\rangle] = \frac{816 + 115\sqrt{29}}{371} \approx 3.8687,$$
$$\lim\big(\alpha\beta^i\big) = \big[(3,3); \langle(3,3,1,1)\rangle\big] + [0; \langle 1,1,3,3\rangle] = \frac{8 + \sqrt{13}}{3} \approx 3.8685.$$

Question. How do the limits of Markov values depend on the Markov sequences? For what Markov sequences α and β is $\lim\big(\alpha^i\beta\big) < \lim\big(\alpha\beta^i\big)$?

Diophantine equation for general Markov numbers

Regular Markov numbers are given by the solutions to the equation

$$x^2 + y^2 + z^2 = 3xyz.$$

Other extensions to the theory of Markov numbers attempt to extend this equation (see [74, 27, 4, 6]).

Question. What Diophantine equation describes all general Markov numbers contained in a tree $\mathcal{T}(\mu, \nu)$?

Markov sequences

We have that the form associated with a Markov sequence attains its Markov minimum at the point $(1, 0)$.

Question. Find general conditions on a non-Markov sequence α for its associated form f_α to attain its Markov minima at a given point.

Bibliography

[1] R. Abe and B. Rittaud. On palindromes with three or four letters associated to the Markoff spectrum. *Discrete Mathematics.*, 340:3032–3043, 2017.

[2] M. Aigner. *Markov's theorem and 100 years of the uniqueness conjecture.* Springer, 2015.

[3] A. Baragar. Asymptotic growth of Markoff-Hurwitz numbers. *Compositio Mathematica.*, 94:1–18, 1994.

[4] A. Baragar. Integral Solutions of Markoff-Hurwitz Equations. *Journal of Number Theory.*, 49:27–44, 1994.

[5] A. Baragar. On the Unicity Conjecture for Markoff Numbers. *Canadian Mathematical Bulletin.*, 39:3–9, 1996.

[6] A. Baragar. The exponent for the Markoff–Hurwitz equations. *Pacific Journal of Mathematics.*, 182:1–21, 1998.

[7] V. Bardakov, V. Shpilrain, and V. Tolstykh. On the palindromic and primitive widths of a free group. *Journal of Algebra.*, 285:574–585, 2005.

[8] J. Berstel and A. de Luca. Sturmian words, Lyndon words and trees. *Theoretical Computer Science.*, 178:171–203, 1997.

[9] J. Berstel, A. Lauve, C. Reutenauer, and F. Saliola. *Combinatorics on Words: Christoffel Words and Repetitions in Words.* CRM Monograph Series, 2008.

[10] A. Bombieri. Continued fractions and the Markoff tree. *Expo. Math.*, 25:187–213, 2007.

[11] Y. Bugeaud and A. Dubickas. Fractional parts of powers and Sturmian words. *C. R. Acad. Sci. Paris, Ser. I.*, 341:69–74, 2005.

[12] Y. Bugeaud and M. Laurent. Exponents of Diophantine Approximation and Sturmian Continued Fractions. *Annales de l'Institut Fourier*, 55:773–804, 2005.

[13] Y. Bugeaud, C. Reutenauer, and S. Siksek. A Sturmian sequence related to the uniqueness conjecture for Markoff numbers. *Theoretical Computer Science.*, 410:2864–2869, 2009.

[14] J. O. Button. The Uniqueness of the Prime Markoff Numbers. *J. London Math. Soc.*, 58:9–17, 1998.

[15] J. O. Button. Markoff Numbers, Principal Ideals and Continued Fraction Expansions. *Journal of Number Theory*, 87:77–95, 2001.

[16] J. W. S. Cassels. The Markoff Chain. *Annals of Mathematics.*, 50:676–685, 1949.

[17] J. W. S Cassels. *An Introduction to Diophantine Approximation.* Cambridge University Press, 1957.

[18] J. W. S Cassels. *An Introduction to the Geometry of Numbers.* Springer, 1991.

[19] E. B. Christoffel. Observatio arithmetica. *Annali di Matematica Pura ed Applicata*, 6(1):148–152, Dec 1873.

[20] H. Cohn. Approach to Markoff's minimal forms through modular functions. *Annals of Math.*, 61:1–12, 1955.

[21] H. Cohn. Representation of Markoff's binary quadratic forms by geodesics on a perforated torus. *Acta Arithmetica.*, 18:125–136, 1971.

[22] H. Cohn. Markoff Forms and Primitive Words. *Acta Arithmetica.*, 196:8–22, 1972.

[23] T. Cusick and M. Flahive. *The Markoff and Lagrange Spectra.* American Mathematical Society, 1989.

[24] T. W. Cusick. The largest gaps in the lower Markoff spectrum. *Duke Mathematical Journal.*, 41:453–463, 1974.

[25] T. W. Cusick. The connection between the Lagrange and Markoff spectra. *Duke Mathematical Journal.*, 42:507–517, 1975.

[26] T. W. Cusick. Endpoints of Gaps in the Markoff Spectrum. *Monatshefte für Mathematik.*, 103:85–91, 1987.

[27] T. W. Cusick. On Perrine's generalized Markoff equations. *Aequationes Mathematicae.*, 46:203—211, 1993.

[28] T. W. Cusick, W. Moran, and A. D. Pollington. Hall's ray in inhomogeneous diophantine approximation. *J. Austral. Math. Soc.*, 60:42–50, 1996.

[29] H. Davenport. On the Minimum of a Ternary Cubic Form. *Journal of the London Mathematical Society.*, 19:13–18, 1944.

[30] H. Davenport. Non-homogeneous ternary quadratic forms. *Acta Mathematica.*, 80:65–95, 1948.

[31] H. Davenport. Note on Indefinite Ternary Quadratic Forms. *Journal of the London Mathematical Society.*, 23:199–202, 1948.

[32] H. Davenport. On Indefinite Ternary Quadratic Forms. *Proceedings of the London Mathematical Society.*, 51:145–160, 1949.

[33] A. de Luca. A combinatorial property of the Fibonacci words. *Information Processing Letters.*, 12:193–195, 1981.

[34] A. de Luca. Sturmian words: structure, combinatorics, and their arithmetics. *Theoretical Computer Science.*, 183:45–82, 1997.

[35] A. de Luca and A. de Luca. Sturmian words and the Stern sequence. *Theoretical Computer Science.*, 581:26–44, 2015.

[36] L. Euler. De fractionibus continuis dissertatio. *Commentarii Academiae Scientiarum Petropolitanae.*, 9:98–137, 1744.

[37] A. Fraenkel, M. Mushkin, and U. Tassa. Determination of $[n\theta]$ by its Sequence of Differences. *Canadian Mathematical Bulletin.*, 21:441–446, 1978.

[38] G. Frobenius. Über die Markoffschen Zahlen. *Preuss. Akad. Wiss. Sitzungsberichte*, pages 458–487, 1913.

[39] E. Galois. Démonstration d'un théorème sur les fractions continues périodiques. *Annales de Gergonne*, 19:294–301, 1828.

[40] A. Glen, J. Justin, S. Widmer, and L. Q. Zamboni. Palindromic richness. *European Journal of Combinatorics.*, 30:510–531, 2009.

[41] A. Glen, A. Lauve, and F. V. Saliola. A note on the Markoff condition and central words. *Information Processing Letters.*, 105:241–244, 2008.

[42] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics.* Addison-Wesley Publishing Company, 1990.

[43] A. Haas. Diophantine approximation on hyperbolic Riemann surfaces. *Acta Mathematica*, 156:33–82, 1986.

[44] M. Hall, Jr. On the Sum and Product of Continued Fractions. *Annals of Mathematics*, 48:966–993, 1947.

[45] H. Helling. A note on the Automorphism Group of the Rank Two Free Group. *Journal of Algebra*, 223:610–614, 2000.

[46] A. Hurwitz. Ueber die angenäherte Darstellung der Irrationalzahlen durch rationale Brüche. *Mathematische Annalen*, 39:279–284, 1891.

[47] B. W. Jones. The Composition of Quadratic Binary Forms. *The American Mathematical Monthly*, 56:380–391, 1949.

[48] B. W. Jones and G. L. Watson. On Indefinite Ternary Quadratic Forms. *Canadian Journal of Mathematics*, 8:592–608, 1956.

[49] O. Karpenkov. Elementary notions of lattice trigonometry. *Mathematica Skandinavica*, 102:161–205, 2008.

[50] O. Karpenkov. On irrational lattice angles. *Functional Analysis and Other Mathematics*, 2:221–239, 2009.

[51] O. Karpenkov. Continued fractions and the second Kepler law. *Manuscripta Mathematica.*, 134:157–169, 2011.

[52] O. Karpenkov. *Geometry of Continued Fractions.* Springer, 2013.

[53] O. Karpenkov and A. Ustinov. Geometry and combinatoric of Minkowski–Voronoi 3-dimensional continued fractions. *Journal of Number Theory*, 176:375–419, 2017.

[54] O. Karpenkov and M. Van Son. Generalized Perron Identity for broken lines. *Journal de théorie des nombres de Bordeaux*, 31:131–144, 2019.

[55] O. Karpenkov and M. van Son. Generalised Markov numbers. *Journal of Number Theory*, 2020.

[56] C. Kassel and C. Reutenauer. Sturmian morphisms, the braid group b_4, Christoffel words and bases of f_2. *Annali di Matematica*, 186:317–339, 2007.

[57] A. Khinchin. *Continued Fractions.* Dover Books on Mathematics, 1964.

[58] F. Klein. Ueber die geometrische Auffassung der gewöhnlichen Kettenbruchentwicklung. *Nachr. Ges. Wiss. Göttingen Math-Phys. Kl.*, pages 357–359, 1895.

[59] A. Korkine and G. Zolotarev. Sur les formes quadratiques. *Mathematische Annalen*, 6:366–389, 1873.

[60] J. L. Lagrange. Additions au mémoire sur la résolution des équations numériques. *Mémoires de l'Académie royale des Sciences et Belles-lettres de Berlin*, XXIV:111–180, 1770.

[61] M. L. Lang and S. P. Tan. A simple proof of the Markoff conjecture for prime powers. *Geom Dedicata*, 129:15–22, 2007.

[62] A. V. Malyshev. Markov and Lagrange spectra (survey of the literature). *Journal of Soviet Mathematics*, 16:767–788, 1981.

[63] A. Markoff. Sur les formes quadratiques binaires indéfinies. *Mathematische Annalen*, 15:381–407, 1879.

[64] A. Markoff. Sur les formes quadratiques binaires indéfinies. (Second mémoire). *Mathematische Annalen*, 17:379–399, 1880.

[65] A. Markoff. Sur une question de Jean Bernoulli. *Mathematische Annalen*, 19:27–36, 1882.

[66] R. A. Mollin. The palindromic index - A measure of ambiguous cycles of reduced ideals without any ambiguous ideals in real quadratic orders. *Journal de théorie des nombres de Bordeaux*, 7:447–460, 1995.

[67] R. A. Mollin. Factorization and palindromic continued fractions. *International Mathematical Forum*, 8:1069–1078, 2013.

[68] R. A. Mollin and K. Cheng. Palindromy and Ambiguous Ideals Revisited. *Journal of Number Theory*, 74:98–110, 1999.

[69] L. J. Mordell. On the Minimum of a Ternary Cubic Form. *Journal of the London Mathematical Society*, 19:6–10, 1944.

[70] L. J. Mordell. The Minimum of a Singular Ternary Quadratic Form. *Journal of the London Mathematical Society*, 2:393–394, 1970.

[71] S. Morier-Genoud and V. Ovsienko. Farey Boat: Continued Fractions and Triangulations, Modular Group and Polygon Dissections. *Jahresber. Dtsch. Math. Ver.*, 121:91–136, 2019.

[72] A. Oppenheim. The lower bounds of indefinite Hermitian quadratic forms. *Quarterly Journal of Mathematics*, 3:10–14, 1932.

[73] R. P. Osborne and H. Zieschang. Primitives in the free group on two generators. *Inventiones mathematicae*, 63:17–24, 1981.

[74] S. Perrine. Sur une généralisation de la Théorie de Markoff. *Journal of Number Theory*, 37:211–230, 1991.

[75] O. Perron. *Die Lehre von den Kettenbrüchen.* B. G. Teubner, 1913.

[76] O. Perron. Über die Approximation irrationaler Zahlen durch rationale, II. *Heidelberger Akademie der Wissenschaften*, 1921.

[77] A. Piggott. Primitives in the free group on two generators. *Journal of Algebra*, 304:359–366, 2006.

[78] C. Reutenauer. On Markoff's property and Sturmian words. *Mathematische Annalen*, 336:1–12, 2006.

[79] C. Reutenauer. Christoffel Words and Markoff Triples. *Integers*, 9:327–332, 2009.

[80] D. Rosen and G. S. Patterson, Jr. Some Numerical Evidence Concerning the Uniqueness of the Markov Numbers. *Mathematics of Computation*, 25:919–921, 1971.

[81] G. Rosenberger. The uniqueness of the Markoff numbers. *Mathematics of Computation*, 30:361–365, 1976.

[82] A. N. Rudakov. The Markov numbers and exceptional bundles on P^2. *Mathematics of the USSR-Izvestiya*, 32:99–112, 1989.

[83] F. Schweiger. *Multidimensional Continued Fractions.* Oxford University Press, 2000.

[84] C. Series. The Geometry of Markoff Numbers. *The Mathematical Intelligencer*, 7:20–29, 1985.

[85] C. Series. The Modular Surface and Continued Fractions. *Journal of the London Mathematical Society*, 31:69–80, 1985.

[86] J. H. Silverman. The Markoff equation $X^2+Y^2+Z^2 = aXYZ$ over quadratic imaginary fields. *Journal of Number Theory*, 35:72–104, 1990.

[87] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences*, (accessed September 18, 2019). `https://oeis.org`.

[88] K. Spalding and A. P. Veselov. Lyapunov spectrum of Markov and Euclid trees. *Nonlinearity*, 30(12):4428–4453, 2017.

[89] B. Springborn. The hyperbolic geometry of Markov's theorem on Diophantine approximation and quadratic forms. *Enseign. Math.*, 63(3–4):333–373, 2017.

[90] A. Srinivasan. Markoff numbers and ambiguous classes. *Journal de théorie des nombres de Bordeaux*, 21:757–770, 2009.

[91] A. Srinivasan. An improvement of the Minkowski bound for real quadratic orders using the Markoff theorem. *Journal of Number Theory*, 131:1420–1428, 2011.

[92] L. Tornheim. Asymmetric minima of quadratic forms and asymmetric Diophantine approximation. *Duke Math. J.*, 22:287–294, 1955.

[93] I. Urbiha. Some properties of a function studied by De Rham, Carlitz and Dijkstra and its relation to the (Eisenstein–) Stern's diatomic sequence. *Math. Commun.*, 6:181–198, 2001.

[94] M. Van Son. Palindromic Sequences of the Markov Spectrum. *Mathematical Notes*, 106:457–467, 2019.

[95] E. M. Wright. Approximation of irrationals by rationals. *The Mathematical Gazette*, 48:288–289, 1964.

[96] B. F. Wyman and M. F. Wyman. An Essay on Continued Fractions. *Mathematical systems theory*, 18:295–328, 1985.

[97] D. Zagier. On the Number of Markoff Numbers Below a Given Bound. *Mathematics of Computation*, 39:709–723, 1982.

[98] Y. Zhang. Congruence and uniqueness of certain Markoff numbers. *Acta Arithmetica*, 128:295–301, 2007.